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Using finite difference method for solving linear two-point fuzzy boundary value problems based on extension principle

S. M. Alavi [a,*]

[a] *Department of Mathematics, Arak Branch, Islamic Azad University, Arak,Iran*

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Abstract

In this paper an efficient Algorithm based on Zadeh's extension principle has been investigated to approximate fuzzy solution of two-point fuzzy boundary value problems, with fuzzy boundary values. We use finite difference method in term of the upper bound and lower bound of r- level of fuzzy boundary values. The proposed approach gives a linear system with crisp tridiagonal coefficients matrix. This linear system determines r-level of fuzzy solution at mesh points. By combining of this solutions, we obtain fuzzy solution of main problem at mesh points, approximately. Its applicability is illustrated by some examples

Key words: Fuzzy differential equation, Zadeh's extension Principle, Finite difference, Fuzzy number, Fuzzy boundary value problems.

1 Introduction

Fuzzy two-point boundary value problems are used to study a variety of problems such as electrostatic, torsion problems,etc. Knowledge about differential equation is often

* Corresponding author's E-mail: m-alavi@iau-arak.ac.ir(Seyed Majid Alavi)

incomplete or vague. For example, parameter values functional relationships, or initial conditions, may not be known precisely. Hence fuzzy differential equation(FDE) was formulated by kaleva [23] and seikkala [36]. Hukuhara derivative of a set valued mapping was first introduced by Hukuhara [23] and it has studied in several works [1,11,23]. Kaleva has formulated fuzzy differential equation(FDE), in term of Hukuhara derivative[23]. Hullermeier,1997, suggested a different formulation of the FIVP based on a family of differential inclusions. Abbasbandy et al[2] represent an easy algorithm for solving one demential FDIs. Numerical methods for FDE are considered in [6,22,29]. Lakshmikantham et al[27] investigated the solution of two point boundary value problems associated with non linear fuzzy differential equations by using the extension principle. The generalized differentiability was introduced by B. Bede and S.G. Gal [7] and modified in [8,9,26,33,5] Allahviranloo et al disused existence and uniqueness of solution to fuzzy deferential equation[3]. In the last few years several published results are proposed to investigate the solution of two-point fuzzy boundary value problems. O'Regan et al. [34] showed that a two-point fuzzy boundary value problems is equivalent to a fuzzy integral equation. Bede [9] show that this statement dos not hold and FBVPs may not have a solution. Chen et al. [13] provided a proof of this statement under certain conditions. Minghao Chen et al.[12] state the existence of solutions to the two-point boundary value problem. Hsuan-Ku Liu [28] investigate the solutions of two-point(FBVPs) as the form $x''(t) = p(t)x'(t) + q(t)x(t) + f(t), x(0) = A$ and $x(l) = B$ where A and B are fuzzy numbers as parametric form, the solution is made by using the lateral type of H-derivative. Nizami et al[32] investigated differential equation with fuzzy boundary value. They have interpreted the two point boundary value problem as a set of crisp problems and have shown that if the solution of the corresponding crisp problem exist and unique solution, then the fuzzy problem has unique solution. Daniel Snchez Ibez et al[35] use the concept of interactivity between fuzzy numbers for the solution to a linear fuzzy boundary value problem (FBVP). They show that a solution of a FBVP, with non-interactive fuzzy numbers as boundary values, can be obtained by the Zadehs Extension Principle. N.A.Gasilov et al[20] present a new approach to a non-homogeneous fuzzy boundary value problem. They consider a linear differential equation with real coefficients but with a fuzzy forcing function and fuzzy boundary values. This paper represent two easy Algorithm for finding fuzzy solution of FBVP by finite difference method. This paper is organized as follows. In section 2 the basic results of the fuzzy numbers and fuzzy calculus are discussed. in section 3 an algorithm is proposed for represent zadeh's Extension principle based on parametric form of fuzzy numbers. In section 4 we study interpretation of two-point crisp boundary value problems. The main methodology is discussed in section 5. In section 6 we apply some easy numerical algorithms for approximate fuzzy solution based on the main methodology that expressed in section 5. In Section 7 some examples are illustrated taht compare exact and approximate solution at some variables.

2 Notation and basic definitions

Definition 2.1 *A fuzzy number is a fuzzy set* $u : \mathbb{R} \to I = [0, 1]$ *which satisfies*
(i) u is upper semicontinuous.
(ii) $u(x) = 0$ *in out side some interval* $[c, d]$.
iii) There are real numbers $a, b : c \leq a \leq\ b \leq d$ *for which*
1.$u(x)$ *is monotonic increasing on* $[c, a]$.
2.$u(x)$ *is monotonic decreasing on* $[b, d]$.
3.$u(x) = 1, a \leq x \leq b$

The set of all such fuzzy numbers is represented by E^1. Fuzzy number with linear sides and the membership function having the following form

$$u(x) = \begin{cases} 0 & if\ x < \alpha_1, \\ \frac{x-\alpha_1}{\alpha_2-\alpha_1} & if\ \alpha_1 \leq x < \alpha_2, \\ 1 & if\ \alpha_2 \leq x \leq \alpha_3, \\ \frac{\alpha_4-x}{\alpha_4-\alpha_3} & if\ \alpha_3 \leq x < \alpha_4, \\ 0 & if\ \alpha_4 < x. \end{cases} \tag{2.1}$$

is called *trapezoidal fuzzy number.*
Since the trapezoidal fuzzy number is completely characterized by four real number $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ it is often denoted in brief as $u(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. A family of all trapezoidal fuzzy number will be denoted by $\mathbb{F}^T$. If $\alpha_2 = \alpha_3$, we obtain *Triangular fuzzy number*

Definition 2.2 *Following [15], we represent arbitrary fuzzy number by an ordered pair of functions* $(\underline{u}(r), \overline{u}(r))$; $0 \leq r \leq 1$ *which satisfy the following requirements,*

1) $\underline{u}(r)$ *is a bounded left continuous non decreasing over* $[0, 1]$
2) $\overline{u}(r)$ *is a bounded left continuous non increasing over* $[0, 1]$
3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Definition 2.3 *The Housdorff distance between two arbitrary fuzzy numbers* $u = (\underline{u}, \overline{u})$ *and* $v = (\underline{v}, \overline{v})$ *is given as*

$$\widetilde{d}_H(u, v) = \sup_{0 \leq r \leq 1} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}. \tag{2.2}$$

3 Extension Principle

The extension principle play a fundamental role in enabling us to extend any point operation to operations involving fuzzy sets. Here for simplicity we focus to 2-place function.

Definition 3.1 *:Let X_1, X_2 and Y are family of sets. Assume f is a mapping from the cartesian product $X_1 \times X_2$ in to Y, that is, for each 2-tuple (t_1, t_2) such that $t_i \in X_i$ we have $f(t_1, t_2) = y \in Y$.*
Let A_1, A_2 are fuzzy sets on X_1, X_2 respectively, then the extension principle allows for the evaluation of $f(A_1, A_2) = B$, where B is a fuzzy subset of Y such that

$$B(y) = \sup\{A_1(t_1) \bigwedge A_2(t_2) | f(t_1, t_2) = y\}.$$

Theorem 3.1 *[31] Let $f : X \times X \to X$ be as continuous function and let A_1 and A_2 be fuzzy numbers. Then*

$$[f(A_1, A_2)]_r = f([A_1]_r, [A_2]_r),$$

where

$$f([A_1]_r, [A_2]_r) = \{f(t_1, t_2) | t_1 \in [A_1]_r, t_2 \in [A_2]_r\}.$$

If $f(x, y)$ is monotonic increasing withe respect to x and monotonic decreasing with respect to y then

$$[f(A, B)]_r = [f(\underline{A}, \overline{B}), f(\overline{A}, \underline{B})].$$

Other cases when f is monotonic is similar.

Corollary 3.1 *Let $f(x, y)$ be linearly with respect to its variables then*

$$[f(A_1, A_2)]_r = [\min\{f(\underline{A}, \underline{B}), f(\underline{A}, \overline{B}), f(\overline{A}, \underline{B}), f(\overline{A}, \overline{B})\}, \max\{f(\underline{A}, \underline{B}), f(\underline{A}, \overline{B}), f(\overline{A}, \underline{B}), f(\overline{A}, \overline{B})\}]. \tag{3.1}$$

4 Two boundary value problems

Consider two-point crisp boundary values problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \\ x(0) = a, \\ x(l) = b. \end{cases} \tag{4.1}$$

where $t \in [0, l]$ and $a,\ b$ are real numbers. The following theorem gives general conditions that the solution to a second order boundary value problem exists and is unique.

Theorem 4.1 *[24] Suppose the function f in the (4.1) is continuous on the set*

$$D_f = \{(t, x, x')|a \leq t \leq b, -\infty \leq x \leq \infty, -\infty \leq x' \leq \infty\},$$

and that the partial derivative f_x and $f_{x'}$ are also continuous on D_f. If

i) $f_x(t, x, x') \geq 0$, *for all* $(t, x, x') \in D_f$, *and*

ii) *a constant M exist, with $|f_{x'}(t, x, x')| \leq M$, for all $(t, x, x') \in D_f$, then (4.1) has a unique solution.*

Corollary 4.1 *If the linear boundary value problem*

$$\begin{cases} x''(t) = p(t)x'(t) + q(t)x(t) + R(t), \\ x(0) = a, \\ x(l) = b. \end{cases} \tag{4.2}$$

satisfies

1. $p(t), q(t)$ *and* $g(t)$ *are continuous on*$[0, l]$,
2. $q(t) > 0$ *on* $[0, l]$,
then the problem has a unique solution

Remark 4.1 *Due to classical text and under hypotheses in Corollary (4.1), let $x_1(t)$ denote the solution to initial value problem*

$$x''(t) = p(t)x'(t) + q(t)x(t) + R(t), \quad 0 \leq t \leq l, \quad x(0) = a, \quad x'(0) = 0,$$

and $x_2(t)$ denote the solution to initial value problem

$$x''(t) = p(t)x'(t) + q(t)x(t), \quad 0 \leq t \leq l, \quad x(0) = 0, \quad x'(0) = 1,$$

then is the unique solution to the linear boundary problem(4.2), provided that, $x_2(l) \neq 0$.

Corollary 4.2 *It is obviously that $x_1(t)$ is a linear function of initial value $x(0) = a$ and $x_2(t)$ is independent of boundary values a, b. Therefore from (**??**), $x(t)$ is a linear function with respect to boundary values a, b and denoted by $x(t) = x(t, a, b)$ such that for a fixed t, $x(t, a, b)$ is linearly withe respect to a, b.*

5 Fuzzy linear two-point boundary value problems

Here we consider a linear differential equation with fuzzy boundary values as follow:

$$\begin{cases} x''(t) = p(t)x'(t) + q(t)x(t) + R(t), \\ x(0) = A, \\ x(l) = B. \end{cases} \tag{5.1}$$

Where $p(t)$,$q(t)$ and $R(t)$ are continuous functions and A, B are fuzzy numbers. Furthermore $x'(t)$ and $x''(t)$ are the usual crisp derivatives of the (crisp) differentiable function $x(t)$ with respect to t.
Let $x : [0, l] \to \mathbb{R}$ is a unique solution to (4.2). According to previous corollary, $x(t)$ can be denoted as $x(t) = x(t, a, b)$. Now we replace a and b by fuzzy numbers $[A]_r = [\underline{A}(r), \overline{A}(r)]$ and $[B]_r = [\underline{B}(r), \overline{B}(r)]$ respectively. Based on extension principe $X(t) = X(t, A, B)$ is fuzzy solution of (5.1) therefore

$$[X(t)]_r = X(t, [A]_r, [B]_r) = [\min\{x(t, a_r, b_r) : (a_r, b_r) \in [A]_r \times [B]_r\}, \max\{x(t, a_r, b_r) : (a_r, b_r) \in [A]_r \times [B]_r\}$$

that is called r-solution of fuzzy boundary value problem (5.1). For an arbitrary ordering pair $(a_r, b_r) \in [A]_r \times [B]_r$ the function $x_r(t) = x(t, a_r, b_r)$ is the unique solution of the

$$\begin{cases} x''_r(t) = p(t)x'_r(t) + q(t)x_r(t) + R(t), \\ x_r(0) = a_r, \\ x_r(l) = b_r. \end{cases} \tag{5.2}$$

Due to corollary (3.1) for a fixed t;

$$\underline{X}(t, r) = \min\{x_r(t, \underline{A}, \underline{B}), x_r(t, \underline{A}, \overline{B}), x_r(t, \overline{A}, \underline{B}), x_r(t, \overline{A}, \overline{B})\}, \tag{5.3}$$

$$\overline{X}(t, r) = \max\{x_r(t, \underline{A}, \underline{B}), x_r(t, \underline{A}, \overline{B}), x_r(t, \overline{A}, \underline{B}), x_r(t, \overline{A}, \overline{B})\}. \tag{5.4}$$

Due to (5.3) and (5.4), r-solution of $X(t)$, that is the set of possible solution of (5.1) at time $t \in [0, l]$, can be determined by solving following system:

$$\begin{cases} x''_r(t) = p(t)x'_r(t) + q(t)x_r(t) + R(t), \\ x_r(0) \in \{\underline{A}, \overline{A}\}, \\ x_r(l) \in \{\underline{B}, \overline{B}\}. \end{cases} \tag{5.5}$$

6 Approximation of Fuzzy solution

In this section we try to apply finite difference method for linear second order value problems

$$\begin{cases} x''_r(t) = p(t)x'_r(t), q(t)x_r(t)) + R(t), \\ x_r(0) = \alpha_r, \\ x_r(l) = \beta_r. \end{cases} \tag{6.1}$$

Where $p(t), q(t)$ and $R(t)$ are continues functions and $\alpha_r \in \{\underline{A}, \overline{A}\}$ and $\beta_r \in \{\underline{B}, \overline{B}\}$. First, we select an integer $n > 0$ and divide the interval $[0, l]$ in to $n+1$ equal subintervals whose end point are the mesh point $t_i = ih$, for $i = 0, 1, 2, \cdots, n+1$, where $h = \frac{l}{n+1}$. The basic question we have to answer in the numerical computation is the follow: Given a $r-$cut of fuzzy boundary values A and B what does the set $X(t_i)$ for $i = 1, 2, \ldots, n$ look like? Our first approximation step is to characterize upper bound of $r-$level set $X(t_i)$, and second is to characterize lower bound of it. Refer to classical text the approximation of $x'_r(t_i)$, $x''_r(t_i)$ are obtained as $x''_r(t_i) \simeq \frac{x_r(t_{i-1}) - 2x_r(t_i) + x_r(t_{i+1})}{h^2}$ and $x'_r(t_i) \simeq \frac{x_r(t_{i+1}) + x_r(t_{i-1})}{2h}$ that are called *Centered-difference formulas*. In equation (6.1) replace Centered-difference formulas to $x'_r(t_i), x''_r(t_i)$ for each $i = 1, 2, \cdots, n$, we have

$$-(1 + \frac{h}{2}p(t_i))w_{r,i-1} + (2 + h^2 q(t_i))w_{r,i} - (1 - \frac{h}{2}p(t_i))w_{r,i+1} = -h^2 R(t_i), \tag{6.2}$$

where $w_{r,0} = \alpha_r$, $w_{r,n+1} = \beta_r$ and $w_{r,i} = w_i(\alpha_r, \beta_r)$ is approximation of $x_r(t_i) = x(t_i, \alpha_r, \beta_r)$, $i = 1, 2, \cdots, n$, .
The coefficient matrix in

$$\begin{cases} d_{ii} = (2 + h^2 q(t_i)) & i = 1, 2, \ldots, n, \\ d_{i,i+1} = -1 + \frac{h}{2}p(t_i) & i = 1, 2, \ldots, n-1, \\ d_{i,i-1} = -1 - \frac{h}{2}p(t_i) & i = 2, 3, \ldots, n. \end{cases} \tag{6.3}$$

Also the right hand side vector in

$$\begin{cases} b_i = -h^2 R(t_i) & i = 2, 3, \ldots n-1, \\ b_1 = -h^2 R(t_1) + (1 + \frac{h}{2}p(t_1))\alpha_r, \\ b_n = -h^2 R(t_n) + (1 - \frac{h}{2}p(t_n))\beta_r. \end{cases} \tag{6.4}$$

Now we recall the theorem that gives conditions under which the tridiagonal linear system (6.4) has a unique solution.

Theorem 6.1 *[24] Suppose that p, q and R are continues on $[a, b]$. If $q(t) \geq 0$ on$[a, b]$, then the tridiagonal linear system (6.2) has a unique solution provided that $h \leq \frac{2}{L}$,* *where* $L = \max_{a \le t \le b} |p(t)|$.

Assume $S = s_{i,j}$ is inverse of coefficient matrix D in (6.2), then we have

$$w_{r,i} = s_{i,1}(-h^2R(t_1)+(1+\frac{h}{2}p(t_1)))\alpha_r+s_{i,n}(-h^2R(t_n)+(1-\frac{h}{2}p(t_n)))\beta_r+\sum_{j=2}^{n-1} s_{i,j}(-h^2R(t_j)). \tag{6.5}$$

It is clear that for each i, $w_{r,i}$ is a linear function of α_r, β_r and denote by $w_{r,i} = w_i(\alpha_r, \beta_r)$, therefor due to previous section the solution of the (**??**) for all t depends upon the α_r β_r as follows:

$$\underline{X}(t_i, r) \simeq \underline{W}_{i,r} = \min\{w_i(\alpha_r, \beta_r) | \alpha_r \in \{\underline{A}, \overline{A}\}, \beta_r \in \{\underline{B}, \overline{B}\}\}, \tag{6.6}$$

$$\overline{X}(t_i, r) \simeq \overline{W}_{i,r} = \max\{w_i(\alpha_r, \beta_r) | \alpha_r \in \{\underline{A}, \overline{A}\}, \beta_r \in \{\underline{B}, \overline{B}\}\}. \tag{6.7}$$

Take $k_{i1} = s_{i,1}(-h^2R(x_1) + (1 + \frac{h}{2}p(x_1)))$ and $k_{in} = s_{i,n}(-h^2R(x_n) + (1 + \frac{h}{2}p(x_n)))$ then (6.5) is equivalent to the following relation.

$$w_{r,i} = w_i(\alpha_r, \beta_r) = k_{i1}\alpha_r + k_{in}\beta_r + \sum_{j=2}^{n-1} s_{i,j}(-h^2R(x_j)). \tag{6.8}$$

we take $k_{i1} = k_{i1}^+ - k_{i1}^-$ where $k_{i1}^+ = \max\{k_{i1}, 0\}$ and $k_{i1}^- = \max\{-k_{i1}, 0\}$. Similarly take $k_{in} = k_{in}^+ - k_{in}^-$ then due to (6.6),(6.7),(6.5) we can state that

$$\underline{W}_{i,r} = k_{i1}^+\underline{A} + k_{in}^+\underline{B} - k_{i1}^-\overline{A} - k_{in}^-\overline{B} + \sum_{j=2}^{n-1} s_{i,j}(-h^2R(x_j)), \tag{6.9}$$

$$\overline{W}_{i,r} = k_{i1}^+\overline{A} + k_{in}^+\overline{B} - k_{i1}^-\underline{A} - k_{in}^-\underline{B} + \sum_{j=2}^{n-1} s_{i,j}(-h^2R(x_j)). \tag{6.10}$$

Proposition 6.1 *Assuming boundary values A and B are fuzzy numbers then W_i for $i = 1, 2, ..., n-1$ given by (6.9) and (6.10) are fuzzy numbers. In addition to that, if A and B are Trapezoidal (Triangular)fuzzy numbers then W_i for $i = 1, 2, ..., n-1$ are Trapezoidal(Triangular) fuzzy numbers.*

Proof. Since $\underline{A}$ and $\underline{B}$ are non decreasing and $\overline{A}$ and $\overline{B}$ are non increasing therefore for any $k \geq 0$, we have

1) $k\underline{A}$ and $k\underline{B}$ are non decreasing,
2) $-k\underline{A}$ and $-k\underline{B}$ are non increasing
3) $k\overline{A}$ and $k\overline{B}$ are non increasing,
4) $-k\overline{A}$ and $-k\overline{B}$ are non decreasing.

In this case we conclude that $\underline{W}'_{i,r}s$ are non decreasing and $\overline{W}'_{i,r}s$ are non increasing. Additionally, for each $i = 1, 2, \ldots, n$,

$$\overline{W}_{i,r} - \underline{W}_{i,r} = k_{i1}^{+}(\overline{A} - \underline{A}) + k_{in}^{+}(\overline{B} - \underline{B}) + k_{i1}^{-}(\overline{A} - \underline{A}) + k_{in}^{-}(\overline{B} - \underline{B}) \geq 0.$$

This show that W_i are fuzzy numbers. Now let A and B are trapezoidal fuzzy numbers then we can take $\underline{A} = a_1 r + a_2$, $\underline{B} = b_1 r + b_2$ where $a_1, b_1 \geq 0$ and $\overline{A} = c_1 r + c_2$, $\overline{B} = d_1 r + d_2$ where $c_1, d_1 \leq 0$

$$\frac{d}{dr}\underline{W}_{i,r} = k_{i1}^{+}a_1 + k_{in}^{+}b_1 - k_{i1}^{-}c_1 - k_{in}^{-}d_1 = const \geq 0$$

and

$$\frac{d}{dr}\overline{W}_{i,r} = k_{i1}^{+}c_1 + k_{in}^{+}d_1 - k_{i1}^{-}c_a - k_{in}^{-}b_1 = const \leq 0$$

then W_i are Trapezoidal(Triangular) fuzzy numbers. Based on (6.9), (6.10) it is necessary to obtain entries of D^{-1}, i.e., $s_{i,j}, i, j = 1, 2, \ldots, n$. Since D is tridiagonal matrix, hence we use factorization Algorithms such as Crout method. Here D has only $(3n-2)$ nonzero entries, there are only $(3n-2)$ conditions to be applied to determine entries of L and U such that $D = LU$, where L is lower triangular and U is upper triangular, i.e., $l_{i,j} = 0$ for $j > i$ and $u_{i,j} = 0$ for $j < i$.

There are $(3n-2)$ undetermined entries of L and U. Let $S_{.,i}$ denote ith column of S and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^t$ that is ith column of identity matrix $n \times n$, then from $DS = I$, we have $DS_{.,i} = e_i$. Therefore entries of S can be found by solving n linear systems, whose coefficient matrices are the same and is tridiagonal. First apply Crout Factorization Algorithm to factor coefficient matrix D as follow: **Algorithm 1** (Crout Factorization Algorithm)
Input(D)

step 1 Set $l_{11} = d_{11}$; $u_{12} = \frac{d_{12}}{l_{11}}$;
step 2 For $i = 2, \ldots, n-1$ set
$l_{i,i-1} = d_{i,i-1}$;
$l_{ii} = d_{ii} - l_{i,i-1}u_{i-1,i}$.
$u_{i,i+1} = \frac{d_{i,i+1}}{l_{ii}}$.
step 3 Set $l_{n,n-1} = d_{n,n-1}$.; $l_{nn} = d_{nn} - \frac{l_{n,n-1}}{u_{n-1,n}}$.

Output The matrices L and U such that $D = LU$.
The factorization of $D = LU$ will be used to finding all columns of D^{-1} and then approximate the solution of fuzzy boundary value problem (5.1). Now we represent Algorithm 2 to approximate fuzzy solutions, for more details see bellow algorithm.
Algorithm 2

INPUT L and U that is obtained from Algorithm 1 and $r \in [0, 1]$.
OUTPUT Approximation of $(\underline{X}(t_1, r), \overline{X}(t_1, r)), \ldots, (\underline{X}(t_n, r), \overline{X}(t_n, r))$.

step 1 For $i = 1, 2, \ldots, n$.
Solve $LZ = e_i$, (e_i is ith column of identity matrix I_n).
Solve $US_{.i} = Z$. (the vectors $S_{.1}, S_{.2}, \ldots, S_{.n}$ are columns of D^{-1} that is denoted by S).

step 2 For $i = 1, 2, \ldots, n$.
Set $k_{i1} = s_{i1}(-h^2R(x_1) + (1 + \frac{h}{2}p(x_1)))$.
Set $k_{in} = b_{i,n}(-h^2R(x_n) + (1 + \frac{h}{2}p(x_n)))$.
$\underline{W}_{i,r} = k_{i1}^+\underline{A} + k_{in}^+\underline{B} - k_{i1}^-\overline{A} - k_{in}^-\overline{B} + \sum_{j=2}^{n-1} s_{i,j}(-h^2R(x_j))$,
$\overline{W}_{i,r} = k_{i1}^+\overline{A} + k_{in}^+\overline{B} - k_{i1}^-\underline{A} - k_{in}^-\underline{B} + \sum_{j=2}^{n-1} s_{i,j}(-h^2R(x_j))$.

step 3 Output $(\underline{W}_{1,r}, \overline{W}_{1,r}), \ldots, (\underline{W}_{n,r}, \overline{W}_{n,r})$

Note that based on proposition (6.1) if A and B are trapezoidal(triangular) fuzzy numbers it is enough to apply algorithm 2 for only $r = 0$ and $r = 1$.

7 Numerical Result

Example 7.1 *Suppose presented method is used to approximate fuzzy solution of*

$$\begin{cases} x''(t) = -\frac{2}{t}x'(t) + \frac{2}{t^2}x(t)) + (2 - \frac{1}{t^2}), \\ x(1) = (0.5r + 1.5, 3 - r), \\ x(3) = (r + 3, 4.5 - 0.5r), \end{cases} \tag{7.1}$$

One can see $p(t) = -\frac{2}{t}$ *and* $L = \max_{1 \le t \le 3} |p(t)| = 2$. *We will use* $n = 8$ *so* $h = 0.25$ *we have* $t_i = 1 + 0.25i$, $i = 0, 1, \ldots, 8$. *According to (6.3) the coefficient matrix* D *can be obtained and due to theorem(6.1) since* $h = 0.25 < \frac{2}{L}$ *then* D^{-1} *exist.*

First Algorithm 1, gives D^{-1} *, then for any* $r \in [0, 1]$ *we can apply algorithm 2 to approximate* $r - cut$ *of fuzzy solution.*

For $r = 0$, *exact solution that is obtained from [32] and approximation solution using Algorithm 2, are listed in Table 1. Also Fig.1 shows exact fuzzy solution and approximate solution at* $t = 2$

Due to definition(2.3), the Housdorff distance between exact and approximation solution at $t = 2$ *is, approximately, 0.0023 i.e.,*

$$\widetilde{d}_H(W_4, X(2)) = 0.0023$$

i	t_i	$k_{i,1}$	$k_{i,2}$	Approx $\underline{X}(t_i,r)$	Approx $\overline{X}(t_i,r)$	Exact $\underline{X}(t_i,r)$	Exact $\overline{X}(t_i)$
0	1.00	–	–	1.5000	3.0000	1.5000	3.0000
1	1.25	0.5533	0.1527	1.1702	2.4143	1.1672	2.4087
2	1.50	0.3633	0.2647	1.0092	2.2556	1.0061	2.2500
3	1.75	0.2449	0.3574	1.1841	2.3304	1.1816	2.3284
4	2.00	0.1670	0.4396	1.3817	2.5659	1.3798	2.5625
5	2.25	0.1070	0.5158	1.6709	2.9156	1.6696	2.9131
6	2.50	0.0632	0.5882	2.0410	3.3615	2.0400	3.3600
7	2.75	0.0285	0.6581	2.4851	3.8920	2.4872	3.8912
8	3.00	–	–	3.0000	4.5000	3.0000	4.5000

Table 1

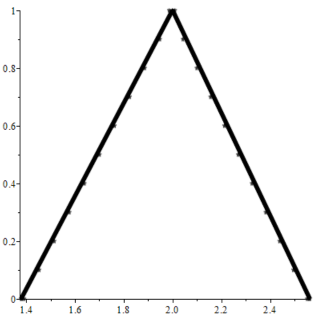

Fig. 1. The approximation(Dots) and Exact(Line) of fuzzy solution at $t = 2$.

Example 7.2 *Approximate fuzzy solution of boundary value problem:*

$$\begin{cases} x''(t) = -16x(t), \\ x(0) = (r-1, 0.5(1-r), \\ x(2) = (0.5(r-1), 0.5(1-r)). \end{cases} \tag{7.2}$$

According to the [32], the exact solution can be presented as follow:

$$X(t,r) = \frac{1}{sin(8)}(sin(8-4t)(-1,0,0.5) + sin(4t)(-0.5,0,0.5)).$$

Now based on presented method take $n = 20$ *so* $h = 0.1$ *and we have* $t_i = 0.1i,\ i = 0, 1, \ldots, 20$. *According to (6.3) the coefficient matrix* D *can be obtained. Since* $p(t_i) = 0$ *then tridiagonal matrix* D *is strictly diagonally dominant therefore* D^{-1} *exist. First Algorithm 1, gives* D^{-1}, *then for any* $r \in [0, 1]$ *we can apply Algorithm.2 to approximate* $r-cut$ *of fuzzy solution. For* $r = 0$, *exact solution that is obtained from [32] and approximation solution using Algorithm 2, are listed in Table.1. Also Fig.2 shows exact fuzzy solution and approximate solution at* $t = 1$

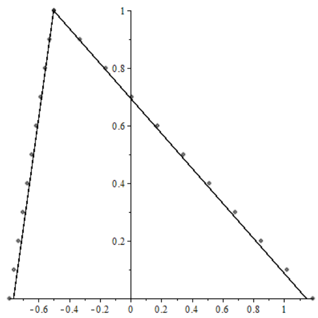

Fig. 2. The approximation(Dots) and Exact(Line) of fuzzy solution at $t = 1$.

i	t_i	$k_{i,1}$	$k_{i,2}$	Approx $\underline{X}(t_i, r)$	Approx $\overline{X}(t_i, r)$	Exact $\underline{X}(t_i, r)$	Exact $\overline{X}(t_i)$
0	0.0	–	–	-1.0000	0.500	-1.0000	0.5000
1	0.1	0.9995	0.6618	-1.1995	0.6995	-1.1751	0.6859
2	0.2	0.8392	0.9109	-1.2071	0.7875	-1.1647	0.7637
3	0.3	0.5446	0.5690	-1.0216	0.7493	-0.9704	0.7207
4	0.4	0.1628	-0.1370	-0.6726	0.5912	-0.6329	0.5640
5	0.5	-0.2449	-0.7554	-0.5834	0.7059	-0.6605	0.7419
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	1.0	-0.7900	-0.7900	-0.7900	1.1851	-0.7649	1.1474
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	1.7	0.9540	0.9540	-1.2263	0.7493	-1.1917	0.7205
18	1.8	0.7358	1.0195	-1.1554	0.7875	-1.1260	0.7636
19	1.9	0.3999	0.9220	-0.8997	0.6997	-0.8827	0.6859
20	2.0	–	–	-0.5000	0.5000	-0.5000	0.5000

Table 2

Due to definition(2.3), the Housdorff distance between exact and approximation solution at $t = 1$ is, approximately,0.027 i.e.,

$$\widetilde{d}_H(W_{10}, X(1)) = 0.027$$

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