Domination Number of Nagata Extension Ring

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Abstract

Let $R$ be a commutative ring with $Z(R)$ the set of zero divisors. The total graph of $R$, denoted by $T(\Gamma(R))$, is the (undirected) graph with all elements of $R$ as vertices, and two distinct vertices are adjacent if their sum is a zero divisor. For a graph $G = (V, E)$, a set $S$ is a dominating set if every vertex in $V \setminus S$ is adjacent to a vertex in $S$. The domination number is equal $|S|$ where $|S|$ is minimum. For $R$-module $M$, an Nagata extension (idealization), denoted by $R(+M)$ is a ring with identity and for two elements $(r, m), (s, n)$ of $R(+M)$ we have $(r, m) + (s, n) = (r + s, m + n)$ and $(r, m)(s, n) = (rs, rn + sm)$. In this paper, we seek to determine the bound for the domination number of total graph $T(\Gamma(R(+M)))$.

Key words: Domination Number, Nagata Extention, Free Torsion $R$-Module, Commutative Ring

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1 Introduction and Preliminaries

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of all dominating sets in $G$. A dominating set with cardinality $\gamma(G)$ is called a $\gamma$-set.

We assume throughout that all rings are commutative with $1 \neq 0$. Let $R$ be a commutative ring with $T(R)$ its total quotient ring, $\text{Reg}(R)$ its set of regular elements, $\text{Z}(R)$ its set of zero divisors, and $\text{Nil}(R)$ its ideal of nilpotent elements. In [5], Anderson and Livingston introduced the zero-divisor graph of $R$, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $\text{Z}(R)^* = \text{Z}(R) \setminus \{0\}$, the set of nonzero zero-divisors of $R$, and for distinct $x, y \in \text{Z}(R)^*$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. This concept is due to Beck [9], who let all the elements of $R$ be vertices and was mainly interested in colorings. For some other recent papers on zero-divisor graphs, see [2,5,7,8,10].

The total graph of $R$, denoted by $T(\Gamma(R))$, as the (undirected) graph with all elements of $R$ as vertices, and for distinct $x, y \in R$, the vertices $x$ and $y$ are adjacent if and only if $x + y \in \text{Z}(R)$. Let $\text{Reg}(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $\text{Reg}(R)$, let $\text{Z}(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $\text{Z}(R)$, and let $\text{Nil}(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ (and $\text{Z}(\Gamma(R))$) with vertices $\text{Nil}(R)$.

Let $G$ be a graph. We say that $G$ is connected if there is a path between any two distinct vertices of $G$. For vertices $x$ and $y$ of $G$, we define $d(x, y)$ to be the length of a shortest path from $x$ to $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path), see [1,3,4].

Recall that for an $R$-module $M$, the idealization of $M$ over $R$ is the commutative ring formed from $R \times M$ by defining addition and multipli-
cation as \((r, m) + (s, n) = (r + s, m + n)\) and \((r, m)(s, n) = (rs, rn + sm)\), respectively. A standard notation for this ”idealized ring” is \(R(+)M\); see [6] for basic properties of rings resulting from the idealization construction. The zero-divisor graph \(\Gamma(R(+)M)\) has recently been studied in [5] and [6].

2 Domination of idealization

Definition 2.1 Let \(R\) be a commutative ring and \(M\) be a \(R\)-module. Idealizer ring \(M\) in \(R\) is denoted by \(R(+)M\) and is defined with two actions addition and multiplication as follows:

\[\begin{align*}
&i) \quad (r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2) \\
&ii) \quad (r_1, m_1) \times (r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)
\end{align*}\]

It is easy to see, \(R(+)M\) with two above actions is a commutative ring.

Definition 2.2 Let \(M\) be a \(R\)-module on commutative ring \(R\). A zero divisor of module \(M\) is defined as follows:

\[Z(M) = \{r \in R : \exists m \in M \text{ s.t. } rm = 0\}\]

Theorem 2.1 Let \(R\) be a commutative ring and \(M\) is a \(R\)-module. Then

\[Z(R(+)M) = Z(R) \times M \cup Z(M) \times M\]

Proof. Suppose \((r, m) \in Z(R(+)M)\), so there is a non-zero \((s, n) \in R(+)M\) such that \((r, m)(s, n) = 0\). Thus, \(rs = 0\) and \(rn + sm = 0\). Now if \(r \in Z(R)\), then the proof is complement. Otherwise \(s = 0\), so \(rn = 0\). Thus, \(r \in Z(M)\). Because \((s, n) \neq 0\) and \(s = 0\), so \(n \neq 0\). Therefore,

\[Z(R(+)M) \subseteq Z(R) \times M \cup Z(M) \times M\]

The proof of other side of inclusion is easy. \(\square\)
Lemma 2.1 Let $x, y$ be adjacent in graph $T(\Gamma(R))$. Then the all members of $A_x$ are adjacent with all members of $A_y$ in graph $T(\Gamma(R(+)M))$, where $A_x = \{(x, m) : m \in M\}$

Proof. Suppose $(x, m) \in A_x$ and $(y, n) \in A_y$. Since $x$ and $y$ are adjacent in graph $T(\Gamma(R))$, so $x + y \in Z(R)$. Therefore, $(x, m) + (y, n) = (x + y, m + n) \in Z(R(+)M)$ and this completes the proof. \[
\]

Lemma 2.2 [12] Let $D = \{(x_i, m_i) : 1 \leq i \leq n\}$ be a set. Then the following are hold.

i) If $D$ is a minimal dominating of $T(\Gamma(R(+)M))$, then for every $i$ and $j$, $x_i \neq x_j$.

ii) If $D$ is a total minimal dominating set of $T(\Gamma(R(+)M))$, then there is a total dominating set $D' = \{(y_i, n_i) : 1 \leq i \leq n\}$ such that for every $i \neq j$, $y_i \neq y_j$.

Theorem 2.2 [12] Let $R$ be a commutative ring and $M$ be a $R$–module. Then

$$\gamma(T(\Gamma(R))) \leq \gamma(T(\Gamma(R(+)M))).$$

If one of the following conditions are established:

i) $M$ be a free torsion $R$–module.

ii) $R = Z(R) \cup U(R)$.

Theorem 2.3 [12] Let $R$ be a commutative ring and $M$ be a $R$–module. Then

$$\gamma_t(T(\Gamma(R(+)M))) \leq \gamma_t(T(\Gamma(R))).$$

Theorem 2.4 [12] Let $R$ be a commutative ring and $M$ be a $R$–module. Then

$$\gamma_t(T(\Gamma(R))) = \gamma_t(T(\Gamma(R(+)M))).$$

If one of the following conditions are established:

i) $M$ be a free torsion $R$–module.

ii) $R = Z(R) \cup U(R)$.
Corollary 2.1 Let $R$ be a finite non-local ring that is not isomorphic with $F \times F \times \cdots \times F$ such that $|F| = 2k + 1$ and $k$ is odd. Also suppose $M$ be a $R$–module. Then
\[ \gamma(t(T(\Gamma(R)))) = \gamma(t(T(\Gamma(R(+)M)))) = \gamma(T(\Gamma(R(+)M))) = \gamma(T(\Gamma(R))) \]

Proof. The results are obtained using the theorems 2.1, 2.2 and 2.2. □

3 Domination and localization

Now, under the new conditions we reduce assumption and find a relation between the following statements.
\[ \gamma_t(T(\Gamma(R))), \gamma_t(T(\Gamma(R(+)M))) \]

Theorem 3.1 Let $R$ be a local ring with maximal ideal $m$ and $|\frac{R}{m}| = k$. Then $\gamma(T(\Gamma(R))) = k$. Moreover, if $\text{char}(R) \neq 2$, then $\gamma_t(T(\Gamma(R))) = k$.

Proof. Suppose $D = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$ is a set of cosets of $m$. We show that $D = \{x_1, x_2, \ldots, x_n\}$ is a dominator set of total graph on $R$.

Let $x \in R$. Then for one index $1 \leq i \leq k$ we have $\overline{x_i} = -\overline{x}$. Equivalently, $x_i + m = -x + m$. Therefore, $x_i + x \in m$. Since $R$ is local, so $m = Z(R)$, i.e. $x$ and $x_i$ are adjacent. Thus, $D$ dominate total graph $T(\Gamma(R))$ and $\gamma(T(\Gamma(R))) \leq k$.

Now, if the set like $D' = \{y_1, y_2, \ldots, y_{k-1}\}$ dominate total graph $T(\Gamma(R))$, then for two distinct index $i, j$, $x_i$ and $x_j$ dominate by only one member of $D'$ like $y_i$. Thus, $x_j + y_i = m_j$ and $x_i + y_i = m_i$ are belong to $m = Z(R)$, as $x_i - x_j = m_i + m_j \in m$, and this is equivalent to $\overline{x_i} = \overline{x_j}$ that is Contradictory with $D$. Therefore, $\gamma(T(\Gamma(R))) = k$.

Finally, if $\text{char}(R) \neq 2$, then for every $i$ there is one $j$ such that $-\overline{x_i} = \overline{x_j}$. So
\[ -x_i + m = x_j + m \Rightarrow x_i + x_j \in m \]
Thats mean the members of $D$ dominate all members of $T(\Gamma(R))$, thus,

$$\gamma_t(T(\Gamma(R))) = \gamma(T(\Gamma(R))) = k.$$ 

\[\square\]

**Definition 3.1** We say that the ring is reduced if there is any non-zero nilpotent member. Equivalently, $R$ is a reduced ring if $x^2 = 0$, then $x = 0$.

**Lemma 3.1** [12] If $R$ is a finite reduced ring, then $R = \prod_{i=1}^{n} F_i$, where for every $1 \leq i \leq n$, $F_i$ is a finite field.

**Theorem 3.2** [12] Let $R$ be a ring but it is not field. Also, suppose $R = \prod_{i=1}^{n} F_i$ ($n \geq 2$), where $F_i$ are field and $|F_1| \leq |F_2| \leq \cdots \leq |F_n|$. Then

$$\gamma(T(\Gamma(R))) = \begin{cases} |F_1| - 1 & R = F_1^2 \text{ and } |R| \text{ is odd} \\ |F_1| & \text{otherwise} \end{cases}$$

moreover, for every ring we have $\gamma_t(T(\Gamma(R))) = |F_1|$.

**Theorem 3.3** [11] Let $R = R_1 \times R_2 \times \cdots \times R_n$, where for every $1 \leq i \leq n$, $(R_i, m_i)$ be local rings and $\frac{R_1}{m_1} = \min \left\{ \frac{R_i}{m_i} : 1 \leq i \leq n \right\}$. If $n \geq 2$ and for at least one $1 \leq k \leq n$, ring $R_k$ is not field, then

$$\gamma(T(\Gamma(R))) = \gamma_t(T(\Gamma(R))) = \left\lfloor \frac{R_k}{m_k} \right\rfloor.$$

**Lemma 3.2** Let $R$ be a commutative ring and $p$ be a prime ideal. Then $Z(R_p) = (Z(R))_p$.

**Proof.** Let $0 \neq \frac{y}{t} \in Z(R_p)$, so there is $\frac{x}{s} \in R_p$ that $\frac{x}{s} \cdot \frac{y}{t} = 0$. Thus, there is $r \in R - p$ such that $rxy = 0$, but $x \neq 0$ and $ry \neq 0$. Otherwise, $\frac{y}{t} = 0$ and $\frac{x}{s} = 0$, that is a contradiction. Therefore, $x \in Z(R)$ and $\frac{y}{t} \in (Z(R))_p$. So we have

$$z(R_p) \subseteq (Z(R))_p$$

On the other, let $0 \neq \frac{y}{t} \in (Z(R))_p$, then $x \in Z(R)$ and $s \in R_p$. So there is $0 \neq \frac{t}{s} \in (Z(R))_p$, thus $x \in Z(R)$ and $s \in R_p$. So there is $0 \neq y \in R$ that $xy = 0$. Now we have $\frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st} = 0$. We show $\frac{y}{t} \neq 0$. Otherwise there is $r \in R - p$ such that $ry = 0$. Since $p$ is prim ideal, so
\[ y \in p, \text{ but } x(r - y) = 0 \text{ and } r - y \in R - p. \text{ Thus,} \]
\[
\frac{x}{s} = \frac{x}{s} \cdot \frac{r - y}{r - y} = \frac{x(r - y)}{s(r - y)} = \frac{0}{s(r - y)} = 0,
\]
that is a contradiction. Therefore, \( \frac{y}{1} \neq 0 \) and this indicates that \( \frac{y}{s} \in Z(R) \) and the proof is complete. \( \square \)

**Lemma 3.3** If \((R, m)\) is local ring, then \( \gamma(T(\Gamma(R))) = \gamma(T(\Gamma(\frac{R}{m}))) \).

**Proof.** Let \( S = \{x_1, x_2, \ldots, x_k\} \) be a \( \gamma \)-set for \( T(\Gamma(R)) \). Then suppose \( S = \{y_1, y_2, \ldots, y_k\} \) and show that this set dominate graph \( T(\Gamma(\frac{R}{m})) \). An arbitrary element in \( \frac{R}{m} \) is form \( \bar{y} \) which \( y \in R \), so there is \( x_j \in S \) such that \( y + x_j \in Z(R) \) and \( y + x_j = y + \bar{x}_j = 0 \). Therefore, \( \bar{y} \) is adjacent \( \bar{x}_j \), i.e. \( S \) dominate \( T(\Gamma(\frac{R}{m})) \), thus,
\[
\gamma(T(\Gamma(R))) \geq \gamma(T(\Gamma(\frac{R}{m}))).
\]
The other side of the inequality is proved to be the same and the equality is established. \( \square \)

**Theorem 3.4** \([6]\) Let \( R \) be a commutative ring, \( I \) a ideal, \( M \) a \( R \)-module and \( N \) be a submodule of \( M \). Then \( I(+)M \) is a ideal of ring \( R(+)M \) iff \( IM \subseteq N \). When \( I(+)M \) is a ideal, then \( \frac{M}{N} \) is a \( \frac{R}{I} \)-module and \( \frac{R(+)M}{I(+)N} = \frac{R}{I}M/I\frac{M}{N} \).

**Theorem 3.5** \([6]\) Let \( R \) be a commutative ring and \( M \) be a \( R \)-module. Maximal ideal of \( R(+)M \) is \( m(+)M \) if \( m \) is maximal ideal of \( R \). Also, ring \( R(+)M \) is quasi-local iff \( R \) be a quasi-local ring. Moreover, \( J(R(+)M) = J(R)(+)M \).

**Theorem 3.6** Let \( R \) be a local ring that not field and \( M \) be a \( R \)-module. Then \( \gamma_{\ell}(T(\Gamma(R))) = \gamma_{\ell}(T(\Gamma(R(+)M))) \).

**Proof.** Let \( m \) be a maximal ideal of \( R \). Then by Theorem 3.5, \( m(+)M \) is a maximal ideal of \( R(+)M \). Also, by Theorem 3.4 we have:
\[
\frac{R(+)M}{m(+)M} = \frac{R}{m} \frac{M}{M} = \frac{R}{m} = \frac{R}{m}.
\]
So $R(+)M$ is local ring. Now, using Lemma 3.3, the proof is completed. \(\Box\)

**Theorem 3.7** Let $R$ be a non-local ring and $p$ be a ideal of $R$. Suppose $R_p$ is a local ring of $R$ with maximal ideal $pR_p$. Then $\gamma_l(T(\Gamma(R))) \leq \gamma_l(T(\Gamma(R_p)))$.

**Proof.** Let $D = \left\{ \frac{x_1}{s_1}, \frac{x_2}{s_2}, \ldots, \frac{x_n}{s_n} \right\}$ be a total dominating set for $R_p$. Without reducing the whole problem can be set $D$ as follows to preserve the domination property:

$$D = \left\{ \frac{x_1}{s}, \frac{x_2}{s}, \ldots, \frac{x_n}{s} \right\}$$

where $s = s_1s_2\ldots s_n$.

We put $y_i = \frac{s_i}{s}x_i$, where $s_i = s_1s_2\ldots s_{i-1}s_{i+1}\ldots s_n$. So we have $\frac{x_i}{s_i} = \frac{y_i}{s}$.

Now, we show that $S = \{y_1, y_2, \ldots, y_n\}$ is a total dominating set for $R$. Suppose $x \in R$. Then $\frac{x}{s} \in R_p$. So there is $y \in \mathbb{Z}(R_p)$ such that $\frac{x+y}{s} = \frac{x}{s} + \frac{y}{s} \in Z(R_p) = (z(R))_p$, as $x + y \in Z(R)$. Therefore, $S$ is a total dominating set for $R$ and the result follows. \(\Box\)

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**References**


