Theory of Approximation and Applications

Vol. 13, No.1, (2019), 1-9

Domination Number of Nagata Extension Ring

Abbas Shariatinia [a,*], Rasoul Mojarad [a]

[a] *Department of Science, Bushehr Branch, Islamic Azad University, Bushehr, Iran.*

Received 12 March 2011; accepted 19 April 2012

Abstract

Let R is a commutative ring whit $Z(R)$ the set of zero divisors. The total graph of R, denoted by $T(\Gamma(R))$ is the (undirected) graph with all elements of R as vertices, and two distinct vertices are adjacent if their sum is a zero divisor. For a graph $G = (V, E)$, a set S is a dominating set if every vertex in $V \setminus S$ is adjacent to a vertex in S. The domination number is equal $|S|$ where $|S|$ is minimum. For R-module M, an Nagata extension (idealization), denoted by $R(+)M$ is a ring with identity and for two elements $(r, m), (s, n)$ of $R(+)M$ we have $(r, m) + (s, n) = (r + s, m + n)$ and $(r, m)(s, n) = (rs, rn + sm)$. In this paper, we seek to determine the bound for the domination number of total graph $T(\Gamma(R(+)M))$.

Key words: Domination Number, Nagata Extention, Free Torsion $R-$ Module, Commutative Ring

2010 AMS Mathematics Subject Classification : 16L99, 16D10

* Corresponding author's E-mail: ashariatinia14@gmail.com

1 Introduction and Preliminaries

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V | \, uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \bigcup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \bigcup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality all of a dominating sets in G. A dominating set with cardinality $\gamma(G)$ is called a $\gamma-$set.

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a commutative ring with $T(R)$ its total quotient ring, $Reg(R)$ its set of regular elements, $Z(R)$ its set of zero divisors, and $Nil(R)$ its ideal of nilpotent elements. In [5], Anderson and Livingston introduced the zero-divisor graph of R, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [9], who let all the elements of R be vertices and was mainly interested in colorings. For some other recent papers on zero-divisor graphs, see [2,5,7,8,10].

The total graph of R, denoted by $T(\Gamma(R))$, as the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. Let $Reg(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $Reg(R)$, let $Z(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $Z(R)$, and let $Nil(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ (and $Z(\Gamma(R))$) with vertices $Nil(R)$.

Let G be a graph. We say that G is connected if there is a path between any two distinct vertices of G. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path), see [1,3,4].

Recall that for an $R-$module M, the idealization of M over R is the commutative ring formed from $R \times M$ by defining addition and multiplication as $(r,m)+(s,n)=(r+s,m+n)$ and $(r,m)(s,n)=(rs,rn+sm)$, respectively. A standard notation for this "idealized ring" is $R(+)M$; see [6] for basic properties of rings resulting from the idealization construction. The zero-divisor graph $\Gamma(R(+)M)$ has recently been studied in [5] and [6].

2 Domination of idealization

Definition 2.1 *Let R be a commutative ring and M be a $R-$module. Idealizer ring M in R is denoted by $R(+)M$ and is defined with two actions addition and multiplication as follows:*

i) $(r_1,m_1)+(r_2,m_2)=(r_1+r_2,m_1+m_2)$

ii) $(r_1,m_1)\times(r_2,m_2)=(r_1r_2,r_1m_2+r_2m_1)$

It is easy to see, $R(+)M$ with two above actions is a commutative ring.

Definition 2.2 *Let M be a $R-$module on commutative ring R. A zero divisor of module M is defined as follows:*

$$Z(M)=\{r\in R:\ \exists m\in M\ s.t.\ rm=0\}$$

Theorem 2.1 *Let R be a commutative ring and M is a $R-$module. Then*

$$Z(R(+)M)=Z(R)\times M\cup Z(M)\times M$$

Proof. Suppose $(r,m)\in Z(R(+)M)$, so there is a non-zero $(s,n)\in R(+)M$ such that $(r,m)(s,n)=0$. Thus, $rs=0$ and $rn+sm=0$. Now if $r\in Z(R)$, then the proof is complement. Otherwise $s=0$, so $rn=0$. Thus, $r\in Z(M)$. Because $(s,n)\neq 0$ and $s=0$, so $n\neq 0$. Therefore,

$$Z(R(+)M\subseteq Z(R)\times M\cup Z(M)\times M$$

The proof of other side of inclusion is easy. □

Lemma 2.1 *Let x, y be adjacent in graph $T(\Gamma(R))$. Then the all members of A_x are adjacent with all members of A_y in graph $T(\Gamma(R(+)M))$, where $A_x = \{(x, m) : \ m \in M\}$*

Proof. Suppose $(x, m) \in A_x$ and $(y, n) \in A_y$. Since x and y are adjacent in graph $T(\Gamma(R))$, so $x + y \in Z(R)$. Therefore, $(x, m) + (y, n) = (x + y, m + n) \in Z(R(+)M)$ and this completes the proof. □

Lemma 2.2 *[12] Let $D = \{(x_i, m_i) : \ 1 \leq i \leq n\}$ be a set. Then the following are hold.*

i) If D is a minimal dominating of $T(\Gamma(R(+)M))$, then for every i and j, $x_i \neq x_j$.

ii) If D is a total minimal dominating set of $T(\Gamma(R(+)M))$, then there is a total dominating set $D' = \{(y_i, n_i) : \ 1 \leq i \leq n\}$ such that for every $i \neq j$, $y_i \neq y_j$.

Theorem 2.2 *[12] Let R be a commutative ring and M be a $R-$module. Then*

$$\gamma(T(\Gamma(R))) \leq \gamma(T(\Gamma(R(+)M))),$$

If one of the following conditions are established:

i) M be a free torsion $R-$ module.

ii) $R = Z(R) \bigcup U(R)$.

Theorem 2.3 *[12] Let R be a commutative ring and M be a $R-$module. Then*

$$\gamma_t(T(\Gamma(R(+)M))) \leq \gamma_t(T(\Gamma(R))).$$

Theorem 2.4 *[12] Let R be a commutative ring and M be a $R-$module. Then*

$$\gamma_t(T(\Gamma(R))) = \gamma_t(T(\Gamma(R(+)M))),$$

If one of the following conditions are established:

i) M be a free torsion $R-$ module.

ii) $R = Z(R) \bigcup U(R)$.

Corollary 2.1 *Let R be a finite non-local ring that is not isomorphic with $F \times F \times \cdots \times F$ such that $|F| = 2k + 1$ and k is odd. Also suppose M be a $R-$module. Then*

$$\gamma_t(T(\Gamma(R))) = \gamma_t(T(\Gamma(R(+)M))) = \gamma(T(\Gamma(R(+)M))) = \gamma(T(\Gamma(R)))$$

Proof. The results are obtained using the theorems 2.1, 2.2 and 2.2. □

3 Domination and localization

Now, under the new conditions we reduce assumption and find a relation between the following statements.

$$\gamma_t(T(\Gamma(R)), \ \gamma_t(T(\Gamma(R(+)M)))$$

Theorem 3.1 *Let R be a local ring with maximal ideal m and $\left|\frac{R}{m}\right| = k$. Then $\gamma(T(\Gamma(R)) = k$. Moreover, if $char(R) \neq 2$, then $\gamma_t(T\Gamma(R)) = k$.*

Proof. Suppose $\overline{D} = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$ is a set of cosets of m. We show that $D = \{x_1, x_2, \ldots, x_n\}$ is a dominator set of total graph on R.

Let $x \in R$. Then for one index $1 \leq i \leq k$ we have $\overline{x_i} = -\overline{x}$. Equivalently, $x_i + m = -x + m$. Therefore, $x_i + x \in m$. Since R is local, so $m = Z(R)$, i. e. x and x_i are adjacent.Thus, D dominate total graph $T(\Gamma(R))$ and $\gamma(T(\Gamma(R))) \leq k$.

Now, if the set like $D' = \{y_1, y_2, \ldots, y_{k-1}\}$ dominate total graph $T(\Gamma(R))$, then for two distinct index i, j, x_i and x_j dominate by only one member of D' like y_t. Thus, $x_j + y_t = m_j$ and $x_i + y_t = m_i$ are belong to $m = Z(R)$, as $x_i - x_j = m_i + m_j \in m$, and this is equivalent to $\overline{x_i} = \overline{x_j}$ that is Contradictory with D. Therefore, $\gamma(T(\Gamma(R))) = k$.

Finally, if $char(R) \neq 2$, then for every i there is one j such that $-\overline{x_i} = \overline{x_j}$. So

$$-x_i + m = x_j + m \ \Rightarrow \ x_i + x_j \in m$$

Thats mean the members of D dominate all members of $T(\Gamma(R))$, thus,

$$\gamma_t(T(\Gamma(R))) = \gamma(T(\Gamma(R))) = k.$$

□

Definition 3.1 *We say that the ring is reduced if there is any non-zero nilpotent member. Equivalently, R is a reduced ring if $x^2 = 0$, then $x = 0$.*

Lemma 3.1 *[12] If R is a finite reduced ring, then $R = \prod_{i=1}^{n} F_i$, where for every $1 \leq i \leq n$, F_i is a finite field.*

Theorem 3.2 *[12] Let R be a ring but it is not field. Also, suppose $R = \prod_{i=1}^{n} F_i$ ($n \geq 2$), where F_i are field and $|F_1| \leq |F_2| \leq \cdots \leq |F_n|$. Then*

$$\gamma(T(\Gamma(R))) = \begin{cases} |F_1| - 1 & R = F_1^n \text{ and } |R| \text{ is odd} \\ |F_1| & \text{otherwise} \end{cases}$$

moreover, for every ring we have $\gamma_t(T(\Gamma(R))) = |F_1|$.

Theorem 3.3 *[11] Let $R = R_1 \times R_2 \times \ldots \times R_n$, where for every $1 \leq i \leq n$, (R_i, m_i) be local rings and $\left|\frac{R_1}{m_1}\right| = min\left\{\left|\frac{R_i}{m_i}\right| : 1 \leq i \leq\right\}$. If $n \geq 2$ and for at least one $1 \leq k \leq n$, ring R_k is not field, then*

$$\gamma(T(\Gamma(R))) = \gamma_t(T(\Gamma(R))) = \left|\frac{R_1}{m_1}\right|.$$

Lemma 3.2 *Let R be a commutative ring and p be a prime ideal. Then $Z(R_p) = (Z(R))_p$.*

Proof. Let $0 \neq \frac{x}{s} \in Z(R_p)$, so there is $\frac{y}{t} \in R_p$ that $\frac{x}{s} \cdot \frac{y}{t} = 0$. Thus, there is $r \in R - p$ such that $rxy = 0$, but $x \neq 0$ and $ry \neq 0$. Otherwise, $\frac{y}{t} = 0$ and $\frac{x}{s} = 0$, that is a contradiction. Therefore, $x \in Z(R)$ and $\frac{x}{s} \in (Z(R))_p$. So we have

$$z(R_p) \subseteq (Z(R))_p$$

On the other, let $0 \neq \frac{x}{s} \in (Z(R))_p$, then $x \in Z(R)$ and $s \in R_p$. So there is $0 \neq y \in R$ that $xy = 0$. Now we have $\frac{x}{s} \cdot \frac{y}{1} = \frac{xy}{s} = 0$. We show $\frac{y}{1} \neq 0$. Otherwise there is $r \in R - p$ such that $ry = 0$. Since p is prim ideal, so

$y \in p$, but $x(r-y) = 0$ and $r - y \in R - p$. Thus,

$$\frac{x}{s} = \frac{x}{s} \cdot \frac{r-y}{r-y} = \frac{x(r-y)}{s(r-y)} = \frac{0}{s(r-y)} = 0,$$

that is a contradiction. Therefore, $\frac{y}{1} \neq 0$ and this indicates that $\frac{x}{s} \in Z(R)$ and the proof is complete. $\square$

Lemma 3.3 *If (R, m) is local ring, then $\gamma(T(\Gamma(R))) = \gamma(T(\Gamma(\frac{R}{m})))$.*

Proof. Let $S = \{x_1, x_2, \cdots, x_k\}$ be a γ-set for $T(\Gamma(R))$. Then suppose $\overline{S} = \{\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_k\}$ and show that this set dominate graph $T(\Gamma(\frac{R}{m}))$. An arbitrary element in $\frac{R}{m}$ is form $\overline{y}$ which $y \in R$, so there is $x_j \in S$ such that $y + x_j \in Z(R)$ and $\overline{y + x_j} = \overline{y} + \overline{x}_j = 0$. Therefore, $\overline{y}$ is adjacent $\overline{x}_j$, i.e. $\overline{S}$ dominate $T(\Gamma(\frac{R}{m}))$, thus,

$$\gamma(T(\Gamma(R))) \geq \gamma(T(\Gamma(\frac{R}{m}))).$$

The other side of the inequality is proved to be the same and the equality is established. $\square$

Theorem 3.4 *[6] Let R be a commutative ring, I a ideal, M a $R-$module and N be a submodule of M. Then $I(+)M$ is a ideal of ring $R(+)M$ iff $IM \subseteq N$. When $I(+)M$ is a ideal, then $\frac{M}{N}$ is a $\frac{R}{I}-$module and $\frac{R(+)M}{I(+)N} = \frac{R}{I}(+)\frac{M}{N}$.*

Theorem 3.5 *[6] Let R be a commutative ring and M be a $R-$module. Maximal ideal of $R(+)M$ is $m(+)M$ if m is maximal ideal of R. Also, ring $R(+)M$ is quasi-local iff R be a quasi-local ring. Moreover, $J(R(+)M) = J(R)(+)M$.*

Theorem 3.6 *Let R be a local ring that not field and M be a $R-$module. Then $\gamma_t(T(\Gamma(R))) = \gamma_t(T(\Gamma(R(+)M)))$.*

Proof. Let m be a maximal ideal of R. Then by Theorem 3.5, $m(+)M$ is a maximal ideal of $R(+)M$. Also, by Theorem 3.4 we have:

$$\frac{R(+)M}{m(+)M} = \frac{R}{m}(+)\frac{M}{M} = \frac{R}{m}(+)0 = \frac{R}{m}.$$

So $R(+)M$ is local ring. Now, using Lemma 3.3, the proof is completed. $\square$

Theorem 3.7 *Let R be a non-local ring and p be a ideal of R. Suppose R_p is a local ring of R with maximal ideal pR_p. Then $\gamma_t(T(\Gamma(R))) \leq \gamma_t(T(\Gamma(R_p)))$.*

Proof. Let $D = \left\{\frac{x_1}{s_1}, \frac{x_2}{s_2}, \dots, \frac{x_n}{s_n}\right\}$ be a total dominating set for R_p. Without reducing the whole problem can be set D as follows to preserve the domination property:

$$D = \left\{\frac{x_1}{s}, \frac{x_2}{s}, \dots, \frac{x_n}{s}\right\}$$

where $s = s_1 s_2 \dots s_n$.

We put $y_i = \overline{s_i} x_i$, where $\overline{s_i} = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_n$. So we have $\frac{x_i}{s_i} = \frac{y_i}{s}$.

Now, we show that $S = \{y_1, y_2, \dots, y_n\}$ is a total dominating set for R. Suppose $x \in R$. Then $\frac{x}{s} \in R_p$. So there is $\frac{y_i}{s}$ such that $\frac{x+y_i}{s} = \frac{x}{s} + \frac{y_i}{s} \in Z(R_p) = (z(R))_p$, as $x + y_i \in Z(R)$. Therefore, S is a total dominating set for R and the result follows. $\square$

Acknowledgment

This research is supported by the research council of the Islamic Azad University of Bushehr.

References

[1] S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, *The total graph and regular graph of a commutative ring*, J. Pure Appl. Algebra, 2009, 213(12), 2224-2228.

[2] S. Akbari, H. R. Maimani and S. Yassemi, When a zero-divisor graph is planar or a complete r-regular graph, J. Algebra, 2003, 270, 169-180

[3] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra, 2008, 320, 2706-2719 Preprint.

[4] D. F. Anderson and A. Badawi, On the total graph of a commutative ring without the zero element, J. Algebra, 2012, 11(4),

[5] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, 1999, 217, 434-447.

[6] D. D. Anderson and Michael Winders, Idealization of a module, J. Algebra, 2009, 1(1), 123-134.

[7] M. Axtell, J. Coykandall and J. Stickles, Zero-divisor graphs of polynomial and power series over commutative rings, Comm. Algebra, 2005, 33, 2043-2050.

[8] M. Axtell and J. Stickles, Zero-divisor graphs of idealizations, J. Pure Appl. Algebra, 2006, 204, 235-243.

[9] I. Beck, Coloring of commutative ring, J. Algebra, 1988, 116(1), 208-226.

[10] T. G. Lucas, The diameter of a zero divisor graph, J. Algebra, 2006, 301, 174-193.

[11] H. R. Maimani C. Wickham, S. Yassemi, Rings whose total graphs have genus at most one, Rocky Mountain J. Math., 2012, 42, 1551-1560.

[12] A. Shariatinia, H. R. Maimani, S. Yassemi, Domination Number Of Total Graphs, Mathematica Slovaca, 2016, 66(6), 34-55.