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General Solution for Fuzzy Linear Second Order Differential Equation Using First Solution

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Abstract

The fuzzy linear second order equations with fuzzy initial values are investigated in this paper. The analytic general solution solutions of them using a first solution is founded. The parametric form of fuzzy numbers is applied to solve the second order equations. General solutions for fuzzy linear second order equations with fuzzy initial values are investigated and formulated in four cases. A example is solved to illustrate method better and solutions are searched in four cases under Hakuhara derivation. Finally the solutions of example are shown in figures for four cases.

Key words: Linear differential equation, Fuzzy initial values, First solution, General solution.

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1 Introduction

The differential equations of second order are apply to solve many problems of applied engineering. The solutions of of these equations are not gotten simply. Thus differential equations of second order are classified in special equations, for example linear and nonlinear equations. One of the linear differential equations is differential equations whit constant multipliers and the Cauchi differential equation is the nonlinear one. These equations are received in the formulation of applied mathematics problems, but in nature the fuzzy second equations are caught for example in physics problems, mechanical problems and etc.
In [9], the H-derivative of fuzzy number-valued function was introduced for solving a fuzzy first order equation then the existence and uniqueness of a solution of fuzzy differential equations were studied in [3,4]. Under H-derivative, the numerical method for solving differential equations is studied by Parandin through Runge-Kutah method, [18]. The strong general differentiable was introduced in [4]. This concept allows us to solve the problem of H-derivative. The existence of fuzzy differential equations of second order are studied by Allahviranloo et al., in [1] and then by Zhang [25], Under the general H-derivative, [16]. Khastan et al. studied solving second order equations under boundary value problem by the general H-derivative, [14]. Allahviranloo and Hooshangian searched fuzzy second order derivations more, investigated the relationships between fuzzy second order derivations and found the solution of fuzzy constant multipliers and Cauchi second order equations with fuzzy initial values [2]. Some numerical methods for solving fuzzy second order derivations are studied in [10,17,23].

In this paper, general derivatives are used to find new solutions for initial value problem of fuzzy linear second order equations with fuzzy initial values. Indeed, with generalized differentiability, the solution for a larger class of them than using H-derivative.

In Section 2, some needed concepts are reviewed. In Section 3, the formula in four cases in order to find the general solutions of linear fuzzy second order differential equations under H-differential are obtained. In section 4, a numerical example is solved in four cases to show more. Finally the conclusion is denoted in section 5.

2 Basic concepts

The basic definitions of a fuzzy number are given in [9,11–13] as follows:

Definition 2.1 *u is named a fuzzy number in parametric form that is shown with a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,*
2. *$\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,*
3. *$\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.*

Definition 2.2 *For arbitrary $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ and $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, $0 \le r \le 1$, and scalar k, it is defined addition, subtraction, scalar product by k and multiplication are respectively as follows:*

$$\text{addition}: \quad \underline{u+v}(r) = \underline{u}(r) + \underline{v}(r), \qquad \overline{u+v}(r) = \overline{u}(r) + \overline{v}(r)$$

$$\text{subtraction}: \quad \underline{u-v}(r) = \underline{u}(r) - \overline{v}(r), \qquad \overline{u-v}(r) = \overline{u}(r) - \underline{v}(r)$$

scalar product:

$$k\tilde{u} = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \ge 0 \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0 \end{cases}$$

$$\text{multiplication}: \quad \underline{uv}(r) = max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$$
$$\overline{uv}(r) = min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$$

Definition 2.3 *Let $u(r) = [\underline{u}(r), \overline{u}(r)], 0 \le r \le 1$ be a fuzzy number, let*

$$u^c = \frac{\underline{u}(r) + \overline{u}(r)}{2}$$

$$u^d = \frac{\overline{u}(r) - \underline{u}(r)}{2}$$

It is clear that $u^d(r) \ge 0$ and $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$

Definition 2.4 *Let $u(r) = [\underline{u}(r), \overline{u}(r)], v(r) = [\underline{v}(r), \overline{v}(r)], 0 \le r \le 1$ are two fuzzy numbers and also k, s are two arbitrary real numbers. If $w = ku + sv$ then*

$$w^c(r) = ku^c(r) + sv^c(r)$$
$$w^d(r) = |k|u^d(r) + |s|v^d(r)$$

Definition 2.5 *Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$ then z is called the H-differential of x, y and it is denoted $x \ominus y$.*

Definition 2.6 [3] *A function $F : I \longrightarrow \mathbb{R}_F$, $I = (a, b)$, is called $H-$differentiable on $t \in I$ if for $h > 0$ sufficiently small there exist the $H-$differences $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and an element $F'(t) \in \mathbb{R}_F$ such that:*

$$0 = lim_{h \to 0} D(\frac{F(t_0 + h) \ominus F(t_0)}{h}, F'(t)) = lim_{h \to 0} D(\frac{F(t_0) \ominus F(t_0 - h)}{h}, F'(t))$$

Definition 2.7 [3] *Let $F : I \to \mathbb{R}_F$ and $t_0 \in I$. F is differentiable at t_0 if there is $F'(t_0) \in \mathbb{R}_F$ such that either*
(i)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0 + h) \ominus F(t_0)$ and $F(t_0) \ominus F(t_0 - h)$ exist and the following limits

$$lim_{h \to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t)$$

or
(ii)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0 + h)$ and $F(t_0 - h) \ominus F(t_0)$ exist and the following limits

$$lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = lim_{h \to 0} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t)$$

or
(iii)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0 + h) \ominus F(t_0)$ and $F(t_0 - h) \ominus F(t_0)$ exist and the following limits

$$lim_{h \to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = lim_{h \to 0} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0)$$

or
(iv)for $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0 + h)$ and $F(t_0) \ominus F(t_0 - h)$ exist and the following limits

$$lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0)$$

Definition 2.8 *Let $F : I \longrightarrow \mathbb{R}_F$. For fix $t_0 \in I$ we say F is differentiable of second-order at t_0, if there is $F''(t_0) \in \mathbb{R}_F$ such that either* *(i)for* $h>0$ *sufficiently close to 0, the H-differences* $F'(t_0+h)\ominus F'(t_0)$ *and* $F'(t_0)\ominus F'(t_0-h)$ *exist and the following limits*

$$lim_{h\to 0}\frac{F'(t_0+h)\ominus F'(t_0)}{h}=lim_{h\to 0}\frac{F'(t_0)\ominus F'(t_0-h)}{h}=F''(t_0)$$

or

(ii)for $h>0$ *sufficiently close to 0, the H-differences* $F'(t_0)\ominus F'(t_0+h)$ *and* $F'(t_0-h)\ominus F'(t_0)$ *exist and the following limits*

$$lim_{h\to 0}\frac{F'(t_0)\ominus F'(t_0+h)}{-h}=lim_{h\to 0}\frac{F'(t_0-h)\ominus F'(t_0)}{-h}=F''(t_0)$$

or

(iii)for $h>0$ *sufficiently close to 0, the H-differences* $F'(t_0+h)\ominus F'(t_0)$ *and* $F'(t_0-h)\ominus F'(t_0)$ *exist and the following limits*

$$lim_{h\to 0}\frac{F'(t_0+h)\ominus F'(t_0)}{h}=lim_{h\to 0}\frac{F'(t_0-h)\ominus F'(t_0)}{-h}=F''(t_0)$$

or

(iv)for $h>0$ *sufficiently close to 0, the H-differences* $F'(t_0)\ominus F'(t_0+h)$ *and* $F'(t_0)\ominus F'(t_0-h)$ *exist and the following limits*

$$lim_{h\to 0}\frac{F'(t_0)\ominus F'(t_0+h)}{-h}=lim_{h\to 0}\frac{F'(t_0)\ominus F'(t_0-h)}{h}=F''(t_0)$$

Theorem 2.1 [1] *If* $f:[a,b]\times\mathbb{R}_F\longrightarrow\mathbb{R}_F$ *is continuous and let* $t_0\in[a,b]$*. A mapping* $x:[a,b]\longrightarrow\mathbb{R}_F$ *is a solution to the initial value problem*

$$x''=f(t,x(t,r),x'(t,r)),\qquad x(t_0)=k_1,\qquad x'(t_0)=k_2$$

if and only if x *and* x' *are continuous and satisfy one of the following conditions:*

(a) $x(t)=k_2(t-t_0)+\int_{t_0}^{t}(\int_{t_0}^{t}f(s,x(s),x'(s))ds)ds+k_1$
where x' and x" are (i)-differentials, or
(b) $x(t)=\ominus(-1)(k_2(t-t_0)\ominus(-1)\int_{t_0}^{t}(\int_{t_0}^{t}f(s,x(s),x'(s))ds)ds)+k_1$
where x' *and* x'' *are (ii)-differentials, or*
(c) $x(t)=\ominus(-1)(k_2(t-t_0)+\int_{t_0}^{t}(\int_{t_0}^{t}f(s,x(s),x'(s))ds)ds)+k_1$

where x' is the (i)-differential and x'' is the (ii)-differential, or
(b) $x(t) = k_2(t - t_0) \ominus (-1)\int_{t_0}^{t}(\int_{t_0}^{t} f(s, x(s), x'(s))ds)ds + k_1$
where x' is the (ii)-differential and x'' is the (i)-differential.

proof: See theorem (3.1) in Ref. [1].□

Theorem 2.2 [3] *Let $[t_0, T] \times E \times E \longrightarrow E$ be continuous and suppose that there exist $M_1, M_2 > 0$ such that*

$$d(f(t, x_1, x_2), f(t, y_1, y_2)) \leq M_1 d(x_1, y_1) + M_2 d(x_2, y_2)$$

for all $t \in [t_0, T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}_F$. Then initial value problem mentioned in theorem (2.9) has a unique solution on $[t_0, T]$ for each case (i) or (ii).

3 General Solution for Fuzzy Linear Second Order Equation

A method for solving fuzzy linear second order equation with fuzzy initial value is studied in this section.
Fuzzy linear second order equation with fuzzy initial values is considered by following:

$$u''(t) + p(t)u'(t) = q(t)u(t), \qquad u(a) = u_0, \qquad u'(a) = u'_0$$

that $p(t)$ and $q(t)$ be two crisp functions. If initial values are fuzzy numbers and there is one given fuzzy solution $u_1(t)$, it can be considered four following cases.

$Case(1):$ If $p(t)$ and $q(t)$ are two crisp functions and positive and u' and u'' are considered (i)-differentiable then:

$$\begin{cases} \overline{u}''(t) + p(t)\overline{u}'(t) = q(t)\overline{u}(t), \\ \underline{u}''(t) + p(t)\underline{u}'(t) = q(t)\underline{u}(t), \\ \underline{u}(a) = \underline{u}_0, \\ \overline{u}(a) = \overline{u}_0, \\ \underline{u}'(a) = \underline{u}'_0, \\ \overline{u}'(a) = \overline{u}'_0 \end{cases} \tag{3.1}$$

Now the following system can be denoted:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t) = \begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}, P(t) = \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix}, Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & q(t) \end{pmatrix}$$

then it can be written the system (1) in the following equation:

$$U''(t) + P(t)U'(t) = Q(t)U(t) \tag{3.2}$$

Now if $U_1(t) = \begin{pmatrix} \overline{u}_1(t) \\ \underline{u}_1(t) \end{pmatrix}$ Then it is considered that the solution of (2) is given:

$$U_2 = U_1 \int (\frac{1}{U_1^2} e^{\int P(t)dt}) dt$$

$Case(2):$ If $p(t)$ and $q(t)$ be two crisp function and are positive and u' and u'' are considered (ii)-differentiable then it can to denote two systems:

$$\begin{cases} \overline{u}''(t) + p(t)\overline{u}'(t) = q(t)\underline{u}(t), \\ \overline{u}(a) = \overline{u}_0, \\ \overline{u}'(a) = \overline{u}'_0 \end{cases} \tag{3.3}$$

and

$$\begin{cases} \underline{u}''(t) + p(t)\underline{u}'(t) = q(t)\overline{u}(t), \\ \underline{u}(a) = \underline{u}_0, \\ \underline{u}'(a) = \underline{u}'_0, \end{cases} \tag{3.4}$$

Now the following system can be denoted:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t)=\begin{pmatrix}\overline{u}(t)\\ \underline{u}(t)\end{pmatrix}, P(t)=\begin{pmatrix}p(t) & 0\\ 0 & p(t)\end{pmatrix}, R(t)=\begin{pmatrix}0 & q(t)\\ q(t) & 0\end{pmatrix}$$

Then it given:

$$U''(t)+P(t)U'(t)=R(t)U(t)$$

Then by using (3) and (4) together it denoted:

$$\begin{cases} u''^{c}(t)+p(t)u'^{c}(t)=q(t)u^{c}(t),\\ u^{c}(a)=u_0^{c},\\ u'^{c}(a)=u'^{c}_{0}, \end{cases} \tag{3.5}$$

and

$$\begin{cases} u''^{d}(t)+p(t)u'^{d}(t)=-q(t)u^{d}(t),\\ u^{d}(a)=u_0^{d},\\ u'^{d}(a)=u'^{d}_{0}, \end{cases} \tag{3.6}$$

Now by solving (5) and (6) together and using case (1):

$$u_2^c=u_1^c\int(\frac{1}{(u_1^c)^2}e^{\int p(t)dt})dt, u_2^d=u_1^d\int(\frac{1}{(u_1^d)^2}e^{\int p(t)dt})dt$$

where $u^c=\frac{\underline{u}+\overline{u}}{2}$, $u'^c=\frac{\underline{u}'+\overline{u}'}{2}$ and $u''^c=\frac{\underline{u}''+\overline{u}''}{2}$ also $u^d=\frac{\overline{u}-\underline{u}}{2}$, $u'^d=\frac{\overline{u}'-\underline{u}'}{2}$ and $u''^d=\frac{\overline{u}''-\underline{u}''}{2}$. If we consider $u_1^c=\frac{\underline{u}_1+\overline{u}_1}{2}$, $u_1^d=\frac{\overline{u}_1-\overline{u}_1}{2}$. Now by definition (2.3) we can find $\underline{u}_2$ and $\overline{u}_2$. Now the general solution is $u(t)=c_1u_1(t)+c_2u_2(t)$ that c_1 and c_2 are two fuzzy numbers.

$Case(3)$: If $p(t)$ and $q(t)$ be two crisp positive function and those are considered that u' is (i)-differentiable and u'' is (ii)-differentiable then it can be denoted two systems: Now the following system can be denoted:

$$\begin{cases} \overline{u}''(t)=p(t)\underline{u}'(t)+q(t)\underline{u}(t),\\ \underline{u}(0)=\underline{u}_0,\\ \overline{u}'(0)=\overline{u}'_0 \end{cases} \tag{3.7}$$

and

$$\begin{cases} \underline{u}''(t) = p(t)\overline{u}'(t) + q(t)\overline{u}(t), \\ \overline{u}(0) = \overline{u}_0, \\ \underline{u}'(0) = \underline{u}'_0, \end{cases} \tag{3.8}$$

The it given the following:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t) = \begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}, S(t) = \begin{pmatrix} 0 & p(t) \\ p(t) & 0 \end{pmatrix}, R(t) = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix}$$

Then the following system is gotten:

$$U''(t) + P(t)U'(t) = R(t)U(t)$$

Now by solving (7) and (8) together we have

$$\begin{cases} u''^c(t) + p(t)u'^c(t) = q(t)u^c(t), \\ u^c(a) = u^c_0, \\ u'^c(a) = u'^c_0, \end{cases} \tag{3.9}$$

and

$$\begin{cases} u''^d(t) - p(t)u'^d(t) = -q(t)u^d(t), \\ u^d(a) = u^d_0, \\ u'^d(a) = u'^d_0, \end{cases} \tag{3.10}$$

Then we have

$$u^c_2 = u^c_1 \int (\frac{1}{(u^c_1)^2} e^{\int p(t)dt})dt, u^d_2 = u^d_1 \int (\frac{1}{(u^d_1)^2} e^{-\int p(t)dt})dt$$

where $u^c = \frac{\underline{u}+\overline{u}}{2}$, $u'^c = \frac{\underline{u}'+\overline{u}'}{2}$, $u''^c = \frac{\underline{u}''+\overline{u}''}{2}$ and $u^d = \frac{\overline{u}-\underline{u}}{2}$ also $u'^d = \frac{\overline{u}'-\underline{u}'}{2}$, $u''^d = \frac{\overline{u}''-\underline{u}''}{2}$ and $u^c_1 = \frac{\underline{u}_1+\overline{u}_1}{2}$, $u^d_1 = \frac{\overline{u}_1-\overline{u}_1}{2}$. Now by definition (2.3) it can be to find $\underline{u}_2$ and $\overline{u}_2$. Thus $u(t) = c_1u_1(t) + c_2u_2(t)$ is the general solution.

$Case(4):$ If $p(t)$ and $q(t)$ be two crisp positive function and these are considered that u' is (ii)-differentiable and u'' is (i)-differentiable then two following systems are denoted:

$$\begin{cases} \overline{u}''(t) = p(t)\underline{u}'(t) + q(t)\underline{u}(t), \\ \underline{u}(0) = \underline{u}_0, \\ \overline{u}'(0) = \overline{u}'_0 \end{cases} \tag{3.11}$$

and

$$\begin{cases} \underline{u}''(t) = p(t)\overline{u}'(t) + q(t)\overline{u}(t), \\ \overline{u}(0) = \overline{u}_0, \\ \underline{u}'(0) = \underline{u}'_0, \end{cases} \tag{3.12}$$

Thus the following system is founded:

$$\begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}'' + \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}' = \begin{pmatrix} 0 & q(t) \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \underline{u} \end{pmatrix}$$

where

$$U(t) = \begin{pmatrix} \overline{u}(t) \\ \underline{u}(t) \end{pmatrix}, S(t) = \begin{pmatrix} 0 & p(t) \\ p(t) & 0 \end{pmatrix}, Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & q(t) \end{pmatrix}$$

Briefly:

$$U''(t) + S(t)U'(t) = Q(t)U(t)$$

Now by solving (11) and (12) together the following equations are given:

$$\begin{cases} u''^c(t) + p(t)u'^c(t) = q(t)u^c(t), \\ u^c(a) = u_0^c, \\ u'^c(a) = u'^c_0, \end{cases} \tag{3.13}$$

and

$$\begin{cases} -u''^d(t) + p(t)u'^d(t) = -q(t)u^d(t), \\ u^d(a) = u_0^d, \\ u'^d(a) = u'^d_0, \end{cases} \tag{3.14}$$

Then we have

$$u_2^c = u_1^c \int (\frac{1}{u_1^c} e^{\int p(t)dt})dt, u_2^d = u_1^d \int (\frac{1}{u_1^d} e^{-\int p(t)dt})dt$$

where $u^c = \frac{\underline{u}+\overline{u}}{2}$, $u'^c = \frac{\underline{u}'+\overline{u}'}{2}$ and $u''^c = \frac{\underline{u}''+\overline{u}''}{2}$ also $u^d = \frac{\overline{u}-\underline{u}}{2}$, $u'^d = \frac{\overline{u}'-\underline{u}'}{2}$, $u''^d = \frac{\overline{u}''-\underline{u}''}{2}$ and $u_1^c = \frac{\underline{u}_1+\overline{u}_1}{2}$, $u_1^d = \frac{\overline{u}_1-\underline{u}_1}{2}$.

Now by definition (2.3) we can find $\underline{u}_2$ and $\overline{u}_2$ and $u(t) = c_1 u_1(t) + c_2 u_2(t)$ is the general solution.

4 Example

Example 4.1 *Consider the following second order differential equation with initial values:*

$$\begin{cases} u''(t) + \frac{1}{t}u'(t) = \frac{1}{t^2}u(t) \\ u(1) = [2\alpha - 1, 3 - 2\alpha] \\ u'(1) = [6\alpha - 5, 5 - 4\alpha] \end{cases} \tag{4.1}$$

The first solution of this equation is $u_1(t) = t$, then four cases it can given in following:
In case (1):
Second solution is $u_2(t) = \frac{-1}{2t}$. The general solution of (15) is:

$$u(t) = [(2\alpha - 3)t + (4\alpha - 2)(-\frac{1}{t}), (4 - 3\alpha)t + (1 - \alpha)(-\frac{1}{t})]$$

The solution of case (1) is drawn in Fig 1.

In case (2):
By using case (2) we have $u_1^d(t) = \frac{1}{t}$, $u_2^d(t) = \frac{t^3}{4}$, $u^c(1) = 1$, $u^d(1) = 1-\alpha$, $u'^c(1) = \alpha$ and $u'^d(1) = 5 - 5\alpha$. The solution of (15) is given by:

$$u(t) = [(\frac{\alpha + 1}{2})t + (\alpha - 1)(\frac{1}{t}) - (6 - 6\alpha)\frac{t^3}{4}, (\frac{\alpha + 1}{2})t + (6 - 6\alpha)\frac{t^3}{4}]$$

The solution of case (2) is drawn in Fig 2.

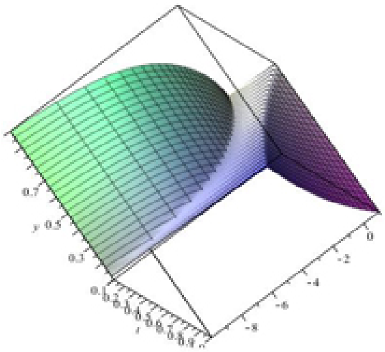

Fig 1: Case (1) in example (4.1)

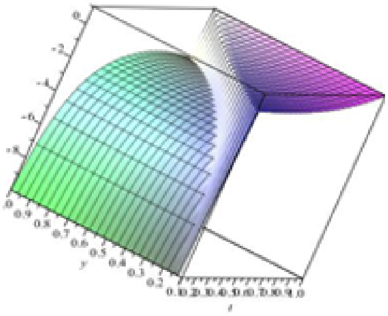

Fig 2: Case (2) in example (4.1)

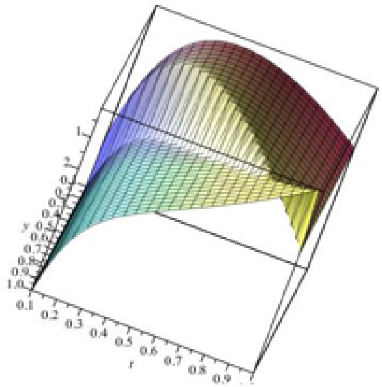

Fig 4: Case (4) in example (4.1)

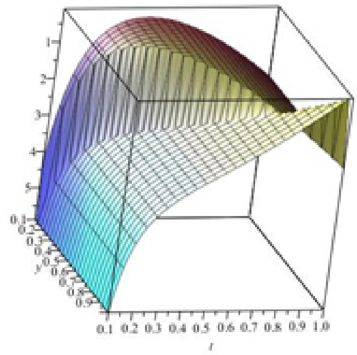

Fig 3: Case (3) in example (4.1)

In case (3):
By using case (3) we have $u_1^c(t) = t$ *and* $u_1^d(t) = t$, $u_2^d(t) = tlnt$, $u^c(1) = 1$, $u^d(1) = 1 - \alpha$, $u'^c(1) = \alpha$ *and* $u'^d(1) = 5 - 5\alpha$. *The solution of (15) is denoted by following:*

$$u(t) = [(\frac{3\alpha - 1}{2})t + (\alpha - 1)(\frac{-1}{2t}) + (4\alpha - 4)tlnt, (\frac{3 - \alpha}{2})t + (\alpha - 1)(\frac{1}{2t}) + (4 - 4\alpha)tlnt]$$

The solution of case (3) is drawn in Fig 3.

In case (4):
By using case (4) we have $u_1^c(t) = t$ *and* $u_1^d(t) = t^{1+\sqrt{2}}$, $u_2^d(t) = t^{\sqrt{2}-1}$, $u^c(1) = 1$, $u^d(1) = 1 - \alpha$, $u'^c(1) = \alpha$ *and* $u'^d(1) = 5 - 5\alpha$. *The solution of (15) is in the following:*
$u(t) = [(\frac{\alpha+1}{2})t + (\alpha-1)(\frac{-1}{2t}) + (\frac{4-\sqrt{2}}{2})(\alpha-1)t^{\sqrt{2}-1} - (\frac{6-\sqrt{2}}{2})(\alpha-1)t^{\sqrt{2}+1}, (\frac{\alpha+1}{2})t +$

$(\alpha-1)(\frac{-1}{2t})-(\frac{4-\sqrt{2}}{2})(\alpha-1)t^{\sqrt{2}-1}+(\frac{6-\sqrt{2}}{2})(\alpha-1)t^{\sqrt{2}+1}]$.
The solution of case (4) is drawn in Fig 4.

5 Conclusion

In this work, the linear fuzzy differential equations of second order with fuzzy initial values considered. Parametric fuzzy number is used to find the general solution with first given solution under Hakuhara derivations in the formula in four cases. Then by using fuzzy initial values four solutions are found. In this paper, the example is solved by method and the solutions for four cases are drawn in Maple 18.

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