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A novel existence and uniqueness theorem for solutions to FDEs driven by Lius process with weak Lipschitz coefficients

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Abstract

This paper we investigate the existence and uniqueness of solutions to fuzzy differential equations driven by Liu's process. For this, it is necessary to provide and prove a new existence and uniqueness theorem for fuzzy differential equations under weak Lipschitz condition. Then the results allows us to consider and analyze solutions to a wide range of nonlinear fuzzy differential equations driven by Liu's process.

Key words: Fuzzy numbers, Fuzzy differential equation, Liu's process, Credibility space condition

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1 Introduction

Most of phenomena and events in the real world occur unexpectedly among which are the changes in economic and political systems, collapse of governments, conflicts between tribes, wars, terrorist attacks. Thus, it is not possible to anticipate or estimate, the price of stocks, valuable papers, monetary units and precious metals accurately. Therefore, the only way find out how this factor can affect the growth or drop in the value of companies is focusing on the price of stocks.

Investigation on effects of the factors along with uncertainty theory can help better understanding and more exact modeling of these phenomena. The *Fuzzy Set* has been presented for the membership function by Zadeh [19] in the beginning of 1965 .

The credibility theory was first introduced by Liu who then presented the concept of credibility measure which is powerful tool for dealing with fuzzy phenomena, to facilitate measuring of fuzzy events that are based on normality, monotonicity, self-duality, and maximality axioms.

Then the concept of fuzzy process was proposed by that introduces a particular fuzzy process with stationary and independent increment named Liu's process which is just like a stochastic process described by Brownian motion.

Since then some literatures has been published on the Liu's process and its applications in other sciences, such as economics and optimal control has been published [20]. Then Liu was inspired by stochastic notions and ito process to introduce fuzzy differential equations [10] which were driven by Liu's process for better understanding the fuzzy phenomena.

In this paper, the following fuzzy differential equation is the considered

$$dx(t) = f(t, x(t)) + g(t, x(t))d\mathbf{C}_t \tag{1.1}$$

where $\mathbf{C}_t$ is a Liu' process, f, g are given functions which satisfy some conditions that we will state later, and $x(t)$ is the solution to the Eq. (1.1) which is in fact a function of a fuzzy process. Regarding to the importance of existence and uniqueness of a solution to fuzzy differential equations driven by Liu's process, Liu investigated the existence and uniqueness of the solution of fuzzy differential equations by employing Lipschitz and Linear growth conditions [18]. Afterward, Fei studied the

uniqueness of solution to the fuzzy differential equations driven by Liu's process with non-Lipschitz coefficients [4].
The main goal of this paper is to provide some weaker conditions to study the existence and uniqueness of solution to the fuzzy differential equations. In this regard, we prove a new existence and uniqueness theorem under the weak Lipschitz condition.
This paper is arranged as follows: Section 2 is devoted to preliminaries on the theory of fuzzy differential equations in the sense of Liu such as credibility theory, credibility measure, credibility space, expected value and variance as well as the definitions of Liu's process, Liu's integral and some necessary inequalities. In section 3, we will focus on the main results including a new existence and uniqueness theorem for the solution of a FDE under a weak condition. This theorem provides us the conditions to deal with some problems that are not previously solvable. Finally, an estimation for the error between approximate solution and accurate solution is given and proven.

2 Preliminaries

Our aim in this section is mainly introducing some concepts such as credibility measure, credibility space, fuzzy variables, independence of fuzzy variables, expected value, variance, fuzzy process, Liu's process, and stopping time.
Assume that Θ is a non-empty set and $\mathcal{P}$ is the power set of Θ. Each element of $\mathbf{A}$ in $\mathcal{P}$ is said to be an event. To provide an axiomatic definition of credibility it is necessary to assign a number $\mathbf{Cr}\{\mathbf{A}\}$ to each event $\mathbf{A}$ which indicates the credibility that $\mathbf{A}$ will occur. Also, in order to ensure that the number $\mathbf{Cr}\{\mathbf{A}\}$ has certain mathematical properties Liu [8]investigate the following four axioms :

(1) Axiom (Normality) $\mathbf{Cr}\{\Theta\} = 1$.
(2) Axiom (Monotonicity) $\mathbf{Cr}\{\mathbf{A}\} \leq \mathbf{Cr}\{\mathbf{B}\}$ whenever $\mathbf{A} \subset \mathbf{B}$.
(3) Axiom (Self-Duality) $\mathbf{Cr}\{\mathbf{A}\} + \mathbf{Cr}\{\mathbf{A}^c\} = 1$ for any event $\mathbf{A}$.
(4) Axiom (Maximality) $\mathbf{Cr}\{\mathrm{U}_i\mathbf{A}_i\} = \sup_i \mathbf{Cr}\{\mathbf{A}_i\}$ for any events

$\{\mathbf{A}_i\}$ with

$$\sup_i \mathbf{Cr}\{\mathbf{A}_i\} < 0.5.$$

Definition 2.1 *[18]. The set function* $\mathbf{Cr}$ *is called a credibility measure which satisfies the normality, monotonicity, self-duality, and maximality axioms.*

Definition 2.2 *[18]. Suppose that* Θ *be a nonempty set,* $\mathcal{P}$ *the power set of* Θ*, and* $\mathbf{Cr}$ *a credibility measure. The triple* $(\Theta, \mathcal{P}, \mathbf{Cr})$ *is called a credibility space.*

Assume that $(\Theta, \mathcal{P}, \mathbf{Cr})$ be a credibility space. A filtration is a family $\{\mathcal{P}_t\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{P}$ (i.e. $\mathcal{P}_t \subset \mathcal{P}_s \subset \mathcal{P}$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $\mathcal{P}_t = \bigcap_{s>t} \mathcal{P}_s$ for all $t \leq 0$. When the credibility space is complete, the filtration is observed to satisfy the usual conditions if it is right continuous and $\mathcal{P}_0$ contains all $\mathbf{Cr}$-null sets.
We also define $\mathcal{P}_\infty = \sigma(U_{t\geq 0}\mathcal{P}_t)$ (i.e. σ-algebra generated by $U_{t\geq 0}\mathcal{P}_t$.)
$\mathcal{P}$-measurable fuzzy variable are determined by $\mathbf{L}^p(\Theta, \mathbf{R}^\mathbf{d})$ that will be defined later.
If the fuzzy variable $x(t)$ is $\mathcal{P}$-measurable for all $t \in [0, t]$, a process is called $\mathcal{P}$-adapted,

Definition 2.3 *[18]. A fuzzy variable is defined as a (measurable) function*
$\xi : (\Theta, \mathcal{P}, \mathbf{Cr}) \longrightarrow \mathbf{R}$.

Definition 2.4 *[18]. The expected value* $\mathbf{E}[\xi]$ *of a fuzzy variable* ξ *is defined by*

$$\mathbf{E}[\xi] = \int_0^{+\infty} \mathbf{Cr}\{\xi \geq r\}\mathrm{dr} - \int_{-\infty}^0 \mathbf{Cr}\{\xi \leq r\}\mathrm{dr}$$

provided that at least one of the two integrals is finite. Besides, the variance is defined by $\mathbf{E}[(\xi - e)^2]$.

Suppose that ξ and η be independent fuzzy variables with finite expected values. Then for any numbers a and b, we have

$$\mathbf{E}[a\xi + b\eta] = a\mathbf{E}[\xi] + b\mathbf{E}[\eta].$$

Definition 2.5 *[10]. Assume that ξ be a fuzzy variable, then the credibility distribution $\mu(x)$ of ξ is defined as follows*

$$\mu(x) = \max\{1, 2\mathbf{Cr}(\xi = x)\}, \quad x \in \mathbf{R}.$$

Definition 2.6 *[10]. A credibility distribution $\mu(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \mu(x) < 1$, and*

$$\lim_{x\to-\infty} \mu(x) = 0, \quad \lim_{x\to+\infty} \mu(x) = 1.$$

Furthermore, the inverse function $\mu^{-1}(\alpha)$ is said to be the inverse credibility distribution of ξ.

Definition 2.7 *[9] A fuzzy process is a function from $\mathbf{T} \times (\Theta, \mathcal{P}, \mathbf{Cr})$ to the set of real numbers where $\mathbf{T}$ is an index and $(\Theta, \mathcal{P}, \mathbf{Cr})$ is a credibility space.*

That means, a fuzzy process $X_t(\theta)$ is a function of two variables such that the function $X_{t^*}(\theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $X_t(\theta^*)$ is said to be a sample path of the fuzzy process. A fuzzy process $X_t(\theta)$ is called sample-continuous if the sample path is continuous for almost all θ.
In this paper, we use the notation $\mathbf{x}(t)$ instead of $\mathbf{x}_t(\theta)$.
A fuzzy process is essentially a sequence of fuzzy variables indexed by time or space. As one of the most important types of fuzzy processes, the Liu's process is defined as follows.

Definition 2.8 *[10]. A fuzzy process $\mathbf{C}_t$ is called a Liu's process if*

(1) $\mathbf{C}_0 = 0$,
(2) $\mathbf{C}_t$ *has stationary and independent increments,*

(3) every increment $\mathbf{C}_{t+s} - \mathbf{C}_s$ is a normally distributed fuzzy variable with expected value **et** *and variance $\sigma^2 t^2$ whose membership function is*

$$\mu(x) = 2(1 + \exp(\tfrac{\pi|x-et|}{\sqrt{6}\sigma t}))^{-1}, \qquad -\infty < x < +\infty.$$

The parameters e and σ are said to be the drift and diffusion coefficients, respectively. Liu's process is called standard if $e = 0$ and $\sigma = 1$.

A fuzzy counterpart of Ito integral is defined in the following based on Liu's process, which is said to be Liu's integral.

Definition 2.9 *[10]. Suppose that $x(t)$ is a fuzzy process and $\mathbf{C}_t$ is a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \ldots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1\leq i\leq k} |t_{i+1} - t_i|.$$

Then the Liu integral of $x(t)$ with respect to $\mathbf{C}_t$ is

$$\int_a^b x(t)d\mathbf{C}_t = \lim_{\Delta\longrightarrow 0} \sum_{i=1}^k x(t_i).(\mathbf{C}_{t_{i+1}} - \mathbf{C}_{t_i})$$

as the limit exists almost surely and is a fuzzy variable.

Theorem 2.1 *[3]. Let $h(t, c)$ be a continuously differentiable function and $\mathbf{C}_t$ be a standard Liu's process. Define $x(t) = h(t, \mathbf{C}_t)$. Then we have the following chain rule*

$$dx(t) = \frac{\partial h}{\partial t}(t, \mathbf{C}_t)dt + \frac{\partial h}{\partial c}(t, \mathbf{C}_t)d\mathbf{C}_t,$$

which is called Liu formula.

Definition 2.10 *[18]. The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if*

$$\mathbf{Cr}\{\bigcap_{i=1}^m \{\xi_i \in \mathbf{B}_i\}\} = \min_{1\leq i\leq m} \mathbf{Cr}\{\xi_i \in \mathbf{B}_i\}$$

for any sets $\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_m$ of real numbers.

Let us define a sequence of credibilistic stopping times.

Definition 2.11 *[9]. A fuzzy variable $\tau : \Theta \to [0, \infty]$ (it may take the value ∞) is said to be an $\{\mathcal{P}_t\}$-stopping time (or simply, stopping time) if $\{\theta : \tau(\theta) \leq t\} \in \mathcal{P}_t$ for any $t \geq 0$*

$$\begin{cases} \tau_k = \inf\{t \geq 0 : |x(t)| \geq k\}, \\ \sigma_1 = \inf\{t \geq 0 : w(x(t)) \geq 2\varepsilon\}, \\ \sigma_{2i} = \inf\{t \geq \sigma_{2i-1} : w(x(t)) \leq \varepsilon\} \quad i = 1, 2, \dots, \\ \sigma_{2i+1} = \inf\{t \geq \sigma_{2i} : w(x(t)) \geq 2\varepsilon\} \ i = 1, 2, \dots, \end{cases}$$

where throughout this paper we set $\inf \phi = \infty$.

Definition 2.12 *[9]. If $X = \{X_t\}_{t \geq 0}$ is a measurable process and τ is a stopping time, then $\{X_{\tau \wedge t}\}_{t \geq 0}$ is said to be a stopped process of X.*

There are several useful inequalities for fuzzy variables, such as Hölder inequality and Chebyshev inequality. In the continuing, we introduce generalized inequalities for fuzzy variables.

Theorem 2.2 *(**Hölder**s Inequality) [4]. Suppose that $\mathbf{p}$ and $\mathbf{q}$ are two positive real numbers with $\frac{1}{\mathbf{p}} + \frac{1}{\mathbf{q}} = 1$, ξ and η be independent fuzzy variables with*

$$\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ] \leq +\infty \quad \textit{and} \quad \mathbf{E}\ [\ |\eta|^{\mathbf{q}}\ ] \leq +\infty.$$

We have

$$\mathbf{E}\ [\ |\xi\eta|\ ] \leq \sqrt[\mathbf{p}]{\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ]}\ \sqrt[\mathbf{q}]{\mathbf{E}\ [\ |\eta|^{\mathbf{q}}\ ]}.$$

Theorem 2.3 *(**Chebychev**'s Inequality). Let $\xi : \theta \to \mathbf{R}^n$ be a fuzzy variable so that $\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ] \leq +\infty$ for some $\mathbf{p}$, $0 \leq \mathbf{p} \leq \infty$.*
Then Chebychev's inequality becomes as the following,

$$\mathbf{Cr}[\ |\xi| \geq \lambda\ ] \leq \frac{1}{\lambda^{\mathbf{p}}}\mathbf{E}\ [\ |\xi|^{\mathbf{p}}\ ] \quad \textit{for all} \quad \lambda \geq 0.$$

Proof. Set $\mathbf{A} = \{w \mid |\xi(w)| \geq \lambda\}$. Then

$$\int_\theta |\xi(w)|^{\mathbf{p}} \mathrm{d}\mathbf{Cr}_w \geq \int_A |\xi(w)|^{\mathbf{p}} \mathrm{d}\mathbf{Cr}_w \geq \lambda^p \mathbf{Cr}_A.$$

Before ending this section it is essential to introduce some symbols that are used in next sections.

Notation 1: $\mathbf{L}^{\mathbf{p}}(\theta, \mathbf{R}^{\mathbf{d}})$ is the family of $\mathbf{R}^{\mathbf{d}}$-valued fuzzy variables ξ with $\mathbf{E}|\xi|^{\mathbf{p}} < \infty$.

Notation 2: $\ell^{\mathbf{p}}([a,b], \mathbf{R}^{\mathbf{d}})$ is the family of $\mathbf{R}^{\mathbf{d}}$-valued $\mathcal{P}_t$-adapted processes $\{f(t)\}_{a \leq t \leq b}$ so that $\int_a^b |f(t)|^{\mathbf{p}} \mathrm{d}t < \infty$ almost surely.

Notation 3: $M^{\mathbf{p}}([a,b], \mathbf{R}^{\mathbf{d}})$ is the family of processes $\{f(t)\}_{a \leq t \leq b}$ in $\ell^{\mathbf{p}}([a,b], \mathbf{R}^{\mathbf{d}})$ so that $\int_a^b |f(t)|^{\mathbf{p}} dt < \infty$.

Notation 4: $\ell^{\mathbf{p}}(\mathbf{R}_+, \mathbf{R}^{\mathbf{d}})$ is the family of processes $\{f(t)\}_{t>0}$ so that for every $T > 0$, $\{f(t)\}_{a \leq t \leq T} \in \ell^{\mathbf{p}}([0,T], \mathbf{R}^{\mathbf{d}})$.

3 Main result

Throughout this paper, we study the fuzzy differential equations

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{C}_t \tag{3.1}$$

where $\mathbf{C}_t$ is a standard Liu's process and f, g are some given functions. $x(t)$ is the solution to the Eq. (3.1) which is a fuzzy process in the sense of Liu.

The Eq. (3.1) is equivalent to the following fuzzy integral equation:

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s)ds + \int_{t_0}^t g(x(s), s)d\mathbf{C}_s. \tag{3.2}$$

In addition, let us state the following conditions.

(**D**) **The Lipshitz condition**: There exists a positive constant $\mathbf{L}$ for all $x(t), y(t) \in \mathbf{R}^d$ and $t \in [t_0, T]$, so that

$$|f(x(t), t) - f(y(t), t)|^2 \vee |g(x(t), t) - g(y(t), t)|^2 \leq \mathbf{L}|x(t) - y(t)|^2.$$

($\mathbf{H}$) **Weak condition**: For $t \in [t_0, T]$ we have

$$f(0,t), g(0,t) \in \mathbf{L}^2[t_0, T]$$

Remark 3.1 *Let coefficient $f(x(t),t)$ and $g(x(t),t)$ of Eq. (3.1) satisfy the conditions ($\mathbf{D}$) and ($\mathbf{H}$). We set $\mathbf{I} = |f(0,t)|^2_{\mathbf{L}^2[0,T]}$, $\mathbf{J} = |g(0,t)|^2_{\mathbf{L}^2[0,T]}$. If $x(t)$ is the solution of Eq. (3.1), then*

$$\mathbf{E}(\sup_{t_0 \le t \le T} |x(t)|^2) \le \mathbf{K}\ e^{6\ \mathbf{L}(T-t_0+1)(T-t_0)}, \tag{3.3}$$

as the $x(t) \in \mathbf{M}^2([t_0,T], \mathbf{R}^d)$, where $\mathbf{K} = (3|x_0|^2 + 6((T-t_0)\mathbf{I} + \mathbf{J}))$.

Theorem 3.1 *Suppose that the conditions ($\mathbf{D}$) and ($\mathbf{H}$) hold. Then the Eq. (3.1) has an unique solution such as $x(t) \in \mathbf{M}^2([t_0,T], \mathbf{R}^n)$ where $x(t)$ is in fact a function of a fuzzy process.*

Proof. Let $x(t)$ and $\overline{x}(t)$ are solutions of Eq.(3.1),
put $\mathbf{a}(w,s) = f(x(s),s) - f(\overline{x}(s),s)$ and $\mathbf{b}(w,s) = g(x(s),s) - g(\overline{x}(s),s)$ where $w \in \theta$. Then

$$x(t) - \overline{x}(t) = \textstyle\int_{t_0}^t \mathbf{a} ds + \int_{t_0}^t \mathbf{b} \mathrm{d}\mathbf{C}(s).$$

By using Lipschitz condition and Holder inequality, we obtain

$$\begin{gathered} |x(t) - \overline{x}(t)|^2 \le 2|\textstyle\int_{t_0}^t \mathbf{a} ds|^2 + 2|\int_{t_0}^t \mathbf{b}\mathrm{d}\mathbf{C}_s|^2 \le \\ 2(t-t_0)\textstyle\int_{t_0}^t \mathbf{L}|x_s - \overline{x}(s)|^2 ds + 2|\int_{t_0}^t \mathbf{b}\mathrm{d}\mathbf{C}_s|^2. \end{gathered}$$

Therefore,

$$\begin{gathered} \sup_{t_0 \le s \le t} |x(s) - \overline{x}(s)|^2 \le \\ 2\mathbf{L}(T-t_0)\textstyle\int_{t_0}^t |x(s) - \overline{x}(s)|^2 ds + 2\sup_{t_0 \le s \le t} |\int_{t_0}^t \mathbf{b}\mathrm{d}\mathbf{C}(s)|^2. \end{gathered}$$

Noting Doob inequality and taking the expectation, we deduce

$$\mathbf{E}(\sup_{t_0 \le s \le t} |x(s) - \overline{x}(s)|^2) \le 2\mathbf{L}(T+4)\textstyle\int_{t_0}^t \mathbf{E}(\sup_{t_0 \le r \le s} |x(r) - \overline{x}(r)|^2) ds.$$

According to Gronwall inequality, we obtain

$$\mathbf{E}(\sup_{t_0 \le t \le T} |x(t) - \overline{x}(t)|^2) = 0. \tag{3.4}$$

Hence $x(t) = \overline{x}(t)$ for all $t_0 \leq t \leq T$ a.s. The uniqueness has been proved. Now, assume that $x^0(t) = x(0), t \in [t_0, T]$, and for $n = 1, 2, \ldots$, define Picard iterations sequence

$$x^n(t) = x(0) + \int_{t_0}^t f(x^{n-1}(s), s)ds + \int_{t_0}^t g(x^{n-1}(s), s)\mathrm{d}\mathbf{C}_s.$$

Obviously, $x^0(0) \in \mathbf{M}^2([0, T], \mathbf{R}^n)$. One can easily verify induction of $x^n(0) \in \mathbf{M}^2([0, T], \mathbf{R}^n)$. According to Holder inequality and using inequality $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$(x^n(t))^2 = 3|x(0)|^2 + 3(t-t_0)\int_{t_0}^t f^2(x^{n-1}(s), s)ds + 3(\int_{t_0}^t g(x^{n-1}(s), s)d\mathbf{C}_s)^2. \tag{3.5}$$

Taking the expectation

$$\mathbf{E}|x^n(t)|^2 = 3\mathbf{E}|x(0)|^2 + 6(T-t_0)\mathbf{E}\int_{t_0}^t [\mathbf{L}|x^{n-1}(s)|^2 + f^2(0, s)]ds + 6\mathbf{E}\int_{t_0}^t [\mathbf{L}|x^{n-1}(s)|^2 + g^2(0, s)]ds$$

$$\leq \mathbf{A} + 6\mathbf{L}[T - t_0 + 1]\int_{t_0}^t \mathbf{E}|x^{n-1}(s)|^2 ds, \tag{3.6}$$

where

$$\mathbf{A} = 3\mathbf{E}|x(0)|^2 + 6[(T-t_0)\mathbf{I} + \mathbf{J}].$$

Due to the Eq. (3.6), for any $k \leq 1$, we have

$$\max_{1\leq n\leq k} \mathbf{E}|x^n(t)|^2 \leq \mathbf{B} + 6\mathbf{L}[T - t_0 + 1]\int_{t_0}^t \max_{1\leq n\leq k} \mathbf{E}|x^n(t)|^2 ds,$$

where

$$\mathbf{B} = \mathbf{A} + 6\ \mathbf{L}\ (T-t_0)(T-t_0+1)\mathbf{E}|x(0)|^2.$$

By using Gronwall inequality for $t_0 \leq t \leq T,\ n \geq 1$, we obtain

$$\max_{1\leq n\leq k} \mathbf{E}|x^n(t)|^2 \leq \mathbf{B}e^{6\ \mathbf{L}\ (T+1)(T-t_0)}, \tag{3.7}$$

noting that

$$|x^1(t) - x^0(t)|^2 \leq 2 \mid \textstyle\int_{t_0}^t f(x(0), s)ds \mid^2 +2 \mid \int_{t_0}^t g(x(0), s)ds \mid^2$$

$$\leq 2(T - t_0) \textstyle\int_{t_0}^t f^2(x(0), s)ds + 2 \mid \int_{t_0}^t g(x(0), s)ds \mid^2.$$

Taking the expectation

$$\mathbf{E}|x^1(t) - x^0(t)|^2 \leq 2(T - t_0)\mathbf{E}(\textstyle\int_{t_0}^t |f(x(s), s)|^2 ds) + 2\mathbf{E}(\int_{t_0}^t |g(x(s), s)|^2 ds)$$

$$\leq 4(T - t_0)\mathbf{E}(\textstyle\int_{t_0}^t (\mathbf{L}|x(0)|^2 + |f(0, s)|^2)ds) + 4\mathbf{E}(\int_{t_0}^t (\mathbf{L}|x(0)|^2 + |g(0, s)|^2)ds)$$

$$\leq 4\mathbf{L}(T - t_0)^2\mathbf{E}(|x(0)|^2) + 4(T - t_0)\mathbf{I} + 4\mathbf{L}(T - t_0)^2\mathbf{E}|x(0)|^2 + 4\mathbf{J} \leq \mathbf{Q}, \quad (3.8)$$

where

$$\mathbf{Q} = 4\mathbf{L}(T - t_0 + 1)(T - t_0)\mathbf{E}(|x(0)|^2) + 4(T - t_0)\mathbf{I} + 4\mathbf{J}.$$

Here we prove that for any $n \geq 0$, we have

$$\mathbf{E}|x^{n+1}(t) - x^n(t)|^2 \leq \frac{\mathbf{Q}[\mathbf{R}(T - t_0)]^n}{n!}, \quad t_0 \leq t \leq T, \qquad (3.9)$$

where $\mathbf{R} = 2\mathbf{L}(T - t_0 + 1)$. From Eq. (3.8), we see that under $n = 0$, Eq. (3.9) holds.

Noting that

$$|x^{n+1}(t) - x^n(t)|^2$$

$$\leq 2 \mid \textstyle\int^t t_0[f(x^n(s), s) - f(x^{n-1}(s), s)]ds \mid^2 +2 \mid \int_{t_0}^t [g(x^n(s), s) - g(x^{n-1}(s), s)]ds \mid^2$$

$$\leq 2\mathbf{L}(T - t_0) \int_{t_0}^t |x^n(s) - x^{n-1}(s)|^2 ds + 2 \mid \int_{t_0}^t [g(x^n(s), s) - g(x^{n-1}(s), s)]ds \mid^2 . \qquad (3.10)$$

Now by **D** condition and taking the expectation, we have

$$E|x^{n+1}(t) - x^n(t)|^2 \leq 2\mathbf{L}(t - t_0 + 1)\mathbf{E} \textstyle\int_{t_0}^t |x^n(s) - x^{n-1}(s)|^2 ds$$

$$\leq 2\mathbf{L}(t-t_0+1)\int_{t_0}^{t}\mathbf{E}|x^n(s)-x^{n-1}(s)|^2ds$$

$$\leq \mathbf{R}\int_{t_0}^{t}\frac{[\mathbf{QR}(t_0-s)]^{n-1}}{(n-1)!}ds=\frac{\mathbf{Q}[\mathbf{R}(T-t_0)]^n}{(n)!}.$$

And we obtain

$$\sup_{t_0\leq t\leq T}|x^{n+1}(t)-x^n(t)|^2\leq 2\mathbf{L}(T-t_0)\int_{t_0}^{t}|x^n(s)-x^{n-1}(s)|^2ds$$

$$+2\sup_{t_0\leq t\leq T}\mid\int_{t_0}^{t}[g(x^n(s),s)-g(x^{n-1}(s),s)]d\mathbf{C}_s\mid^2.$$

Also, by taking the expectation

$$\mathbf{E}(\sup_{t_0\leq t\leq T}\mathbf{E}|x^{n+1}(t)-x^n(t)|^2)\leq 2\mathbf{L}(T-t_0+1)\int_{t_0}^{t}\mathbf{E}|x^{n+1}(s)-x^n(s)|^2$$

$$\leq 4\mathbf{R}\int_{t_0}^{t}\frac{\mathbf{Q}[\mathbf{R}(t_0-s)]^{n-1}}{(n-1)!}ds=\frac{\mathbf{Q}[\mathbf{R}(T-t_0)]^n}{(n)!}.$$

Using the Chebyshev's equality, we derive

$$\mathbf{Cr}\{\sup_{t_0\leq t\leq T}\mathbf{E}|x^{n+1}(t)-x^n(t)|>\frac{1}{2^n}\}\leq\frac{4\mathbf{Q}[\mathbf{R}(T-t_0)]}{n!},$$

since

$$\sum_{n=0}^{\infty}\frac{4\mathbf{Q}[4\mathbf{R}(T-t_0)]^n}{n!}<\infty,$$

by Borel-Cantell lemma, for almost all for $\omega\in\theta$. There is a positive integer
$n_0=n_0(\omega)$, so that $n\geq n_0$, we have

$$\sup_{t_0\leq t\leq T}|x^{n+1}(t)-x^n(t)|\leq\frac{1}{2^n}.$$

We know that the partial sums

$$x^0(t)+\sum_{i=0}^{n-1}[x^{i+1}(t)-x^i(t)]=x^n(t)$$

are uniformly in $t\in[0,T]$. It is clear, that $x(t)$ is continuous and $\mathcal{P}_t$ is adapted. On the other hand, from Eq. (3.9), the statement $\{x^n(t)\}_{n\leq 1}$ is a Cauchy in $\mathbf{L}^2$ for every t. Therefore $x(t)\in\mathbf{L}^2[0,T]$ in Eq. (3.7). Let $n\to\infty$ in Eq. (3.6) Then,

$$\mathbf{E}|x(t)|^2\leq\mathbf{B}e^{6\mathbf{L}(T+1)(T-t_0)}, t_0\leq t\leq T.$$

Therefore, $x(t) \in \mathbf{M}^2([t_0, T], \mathbf{R}^d)$. We deduce that $x(t)$ satisfies Eq. (3.1). Note that $(n \to \infty)$, so we have

$$E \mid \int_{t_0}^t f(x^n(s), s)ds - \int_{t_0}^t f(x(s), s)ds \mid^2 + \mathbf{E} \mid \int_{t_0}^t g(x^n(s), s)ds - \int_{t_0}^t g(x(s), s)ds \mid^2$$

$$\leq \mathbf{L}(T - t_0 + 1) \int_{t_0}^t \mathbf{E}|x^n(s) - x(s)|^2 ds \to 0.$$

Then in Eq. (3.5), letting $n \to \infty, t_0 \leq t \leq T$. We have

$$x(t) = x(0) + \int_{t_0}^t f(x(s), s)ds + \int_{t_0}^t g(x(s), s)d\mathbf{C}_s.$$

The following theorem gives an estimate of solution.

Theorem 3.2 *Assume that the coefficients $f(x(t), t)$ and $g(x(t), t)$ satisfy $\mathbf{D}$ conditions, then $x(t)$ is the unique solution to Eq. (3.1), and $x^n(t)$ be the iterations defined by Eq. (3.5), then*

$$(\sup_{t_0 \leq t \leq T} |x^n(t) - x(t)|^2) \leq \frac{8\mathbf{Q}[\mathbf{R}(T - t_0)]^n}{n!} e^{8\mathbf{R}(T-t_0)}, \tag{3.11}$$

for all $n \geq 1$, where $\mathbf{Q}$ and $\mathbf{R}$ are the same as defined in the proof of Theorem 3.2, that is

$$\mathbf{Q} = 4\mathbf{L}(T - t_0)(T - t_0 + 1)\mathbf{E}(|x(0)|^2) + 4(T - t_0)\mathbf{I} + 4\mathbf{J}.$$

Proof. From

$$x^n(t) - x(t) = \int_{t_0}^t [f(x^{n-1}(s), s)ds - f(x(s), s)]ds$$

$$\int_{t_0}^t [g(x^{n-1}(s), s)ds - g(x(s), s)]ds,$$

we derive that

$$\mathbf{E}(\sup_{t_0 \leq s \leq t} |x^n(t) - x(t)|^2)$$

$$\leq 2\mathbf{L}(T - t_0 + 2) \int_{t_0}^t \mathbf{E}|x^{n-1}(s) - x(s)|^2$$

$$\leq 8\mathbf{R} \int_{t_0}^t \mathbf{E}|x^n(s) - x^{n-1}(s)|^2 + 8\mathbf{R} \int_{t_0}^t \mathbf{E}|x^n(s) - x(s)|^2.$$

Replacing Eq. (3.9) in to this, we get the following result

$$\mathbf{E}(\sup_{t_0\le s\le t}|x^n(s)-x(s)|^2)$$

$$\le 8\mathbf{R}(\int_{t_0}^{T}\frac{\mathbf{Q}[\mathbf{R}(t_0-s)]^{n-1}}{(n-1)!}ds+\int_{t_0}^{t}\mathbf{E}|x^n(s)-x(s)|^2ds)$$

$$\le \frac{8\mathbf{Q}[\mathbf{R}(T-t_0)]^{n}}{n!}+8\mathbf{R}\int_{t_0}^{t}\mathbf{E}(\sup_{t_0\le r\le s}|x^n(r)-x(r)|^2)ds.$$

Consequently, by using Gronwall inequality Eq. (3.11), the proof is completed.

References

[1] X. Chen, X. Qin. A new existence and uniqueness theorem for fuzzy differential equations. International Journal of Fuzzy Systems., 2013, 13(2): 148 - 151.

[2] Z. Ding, M. Ma, A. Kandel. Existence of the solutions of fuzzy differential equations with parameters. J. Information Sciences., 1999, 99(1): 1205 - 1217.

[3] W. Dai. Lipschitz continuity of Liu process. Proceedings of the Eighth International Conference on Information and Management Science China, 2009: 756 - 760.

[4] W. Fei. Uniqueness of solutions to fuzzy differential equations driven by Lius process with non-Lipschitz coefficients. Internatinal Confrance On Fuzzy and Knowledge Discovery, 2009: 565 - 569.

[5] M. Friedman, M. Ming, A. Kandel. Numerical procedures for solving fuzzy differential and integral equations. Internat Confrance Fuzzy Logic and Applications, Israel, 1997: 18 - 21.

[6] J. Gao. Credibilistic option pricing. Journal of Uncertain Systems, 2008, 2(4): 243 - 247.

[7] O. Kaleva. Fuzzy differential equation. J. Fuzzy Sets and Systems, 1987, 24(2): 301 - 317.

[8] X. Li, B. Liu. A suffcient and necessary condition for credibility measures. International Journal of Uncertainty, Fuzziness and Knowledge Based Systems, 2004, 14(5): 527 - 535.

[9] B. Liu. Uncertainty Theory. Springer-Verlag, Berlin, 2004.

[10] B. Liu. A survey of credibility theory. J. Fuzzy Optimization and Decision Making, 2006, 5(4): 387 - 408.

[11] B. Liu. Uncertainty Theory, 2nd Ed. Springer-Verlag. Berlin, 2007.

[12] B. Liu. Fuzzy process, hybrid process and uncertain process. Journal of Uncertain Systems, 2008, 2(1): 3 - 16.

[13] B. Liu, Y. Liu. Expected value of fuzzy variable and fuzzy expected value models. IEEE Transactions on Fuzzy Systems, 2002, 10(4): 445 - 450.

[14] J. Peng. A general stock model for fuzzy markets. Journal of Uncertain Systems, 2008, 2(4): 248 - 254.

[15] Z. Qin, X. Li. Option pricing formula for fuzzy financial market. Journal of Uncertain Systems, 2008, 2(1): 17 - 21.

[16] Z. Qin, X. Gao. Fractional Liu process with application to finance. J. Mathematical and Computer Modeling, 2009, 50(9): 1538 - 1543.

[17] C. You. Multi-dimensional Liu process, differential and integral. Proceedings of the First Intelligent Computing Conference. China., 2007: 153 - 158.

[18] C. You, W. Wang, H. Huo. Existence and uniqueness theorems for fuzzy differential equations. Journal of Uncertain Systems, 2013, 7(4): 303 - 315.

[19] L.A. Zadeh. Fuzzy sets. J. Information and Control, 1965, 8(1): 338 - 353.

[20] Y. Zhu. A Fuzzy option control with application to portfolio selection. Journal of Uncertain Systems, 2009, 3(4): 270 - 279.