

Viscosity Approximation Methods for W-mappings in Hilbert space

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Abstract

We suggest a explicit viscosity iterative algorithm for finding a common element of the set of common fixed points for W-mappings which solves some variational inequality. Also, we prove a strong convergence theorem with some control conditions. Finally, we apply our results to solve the equilibrium problems. Finally, examples and numerical results are also given.

Key words: Nonexpansive mapping, equilibrium problems, strongly positive linear bounded operator, fixed point, Hilbert space, W-mapping.

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1 Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H . A mapping T of C into itself is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$, we denote $F(T)$ the set of fix points of T . The strong(weak) convergence of $\{x_n\}$ to x is written by $x_n \rightarrow x$ ($x_n \rightharpoonup x$) as $n \rightarrow \infty$.

For any $x \in H$, there exists a unique nearest point in C , denoted it by $P_C(x)$ such that

$$\|x - P_C x\| \leq \|x - y\|, \text{ for all } y \in C,$$

such that a mapping P_C from H onto C is called the *metric projection*. Recall that H satisfies the Opial's condition [6] if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$. A self mapping $f : C \rightarrow C$ is a contraction if there exists $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for each $x, y \in C$.

An operator A is said to be a strongly positive linear bounded operator on H , if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \text{ for all } x \in H.$$

Let F be a bifunction of $C \times C$ into R . The equilibrium problems for $C \times C \rightarrow C$, is to find $x \in C$ such that

$$F(x, y) \geq 0, \text{ for all } y \in C. \tag{1.1}$$

The set of solution of Eq.(1.1) is denoted by $EP(F)$. Several problems in physics, optimization, and economics reduce to find a solution of Eq.(1.1)

[1], [4]. We consider the following iteration [10]

$$\begin{aligned}
U_{n,n+1} &:= I, \\
U_{n,n} &:= \lambda_n S_n U_{n,n+1} + (1 - \lambda_n)I, \\
U_{n,n-1} &:= \lambda_{n-1} S_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\
&\vdots \\
U_{n,k} &:= \lambda_k S_k U_{n,k+1} + (1 - \lambda_k)I, \\
&\vdots \\
U_{n,2} &:= \lambda_2 S_2 U_{n,3} + (1 - \lambda_2)I, \\
W_n = U_{n,1} &:= \lambda_1 S_1 U_{n,2} + (1 - \lambda_1)I,
\end{aligned} \tag{1.2}$$

where $\lambda_1, \lambda_2, \dots$ are real numbers such that $0 \leq \lambda_n \leq 1$, and S_1, S_2, \dots be an infinite nonexpansive mappings. It is clear that nonexpansivity of each S_n ensure the nonexpansivity of W_n . Such a mapping W_n is called W -mapping generated by S_n, S_{n-1}, \dots, S_1 and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$.

In this paper, by intuition from [7], a new iterative scheme is introduced. This scheme find a common solution of the equilibrium problem (EP) and fixed point problem for an infinite family of nonexpansive mappings. Also, we prove a strong convergence theorem.

The following lemmas will be useful for proving the main results of this article:

Lemma 1.1 [8] *Let C be a nonempty closed convex subset of a Banach space E and $\{S_n\} : C \rightarrow C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. For any $n \geq 1$, let W_n be the W -mapping of C into itself generated by S_n, S_{n-1}, \dots, S_1 and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strictly convex, then $F(W_n) = \bigcap_{i=1}^n F(S_i)$.*

Lemma 1.2 [8] *Let C be a nonempty closed convex subset of a strictly convex Banach space E . Let $\{S_n\} : C \rightarrow C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$*

and $k \geq 1$ $\lim_{n \rightarrow \infty} U_{n,k}x$ exists.

Lemma 1.3 [8] Let C be a nonempty closed convex subset of strictly convex Banach E . Let $\{S_n\} : C \rightarrow C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then W is a nonexpansive mapping and $F(W) = \bigcap_{n=1}^{\infty} F(S_n)$.

Lemma 1.4 [2] Let C be a nonempty closed convex subset of a Hilbert space H and $\{S_n\} : C \rightarrow C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. If K is any bounded subset of C , then $\limsup_{n \rightarrow \infty} \|Wx - W_n x\| = 0$.

Lemma 1.5 [5] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq I - \rho \bar{\gamma}$.

Lemma 1.6 [9] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 1.7 [2] Let H be a real Hilbert space. Then the following holds:

- (a) $\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y \rangle$ for all $x, y \in H$,
- (b) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$,
- (c) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$.

Lemma 1.8 [1] Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into R satisfying the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in K$,
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$,
- (A3) for each $x, y, z \in K$

$$\lim_{t \rightarrow 0} F(tz + (1 - t)x, y) \leq F(x, y),$$

(A4) for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semi-continuous. Let $r > 0$ and $x \in H$.

Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in K.$$

Lemma 1.9 [3] Let K be a nonempty closed convex subset of H and let F be a bifunction of $K \times K$ into \mathbb{R} satisfying (A1) – (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow K$ as follows:

$$T_r(x) = \{z \in K : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in K\},$$

for all $x \in H$. Then the following hold

- (i) T_r is single valued map,
- (ii) T_r is firmly nonexpansive, that is, for any $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle,$$

- (iii) $F(T_r) = EP(F)$,
- (iv) $EP(F)$ is closed and convex.

Lemma 1.10 [11] Assume $\{a_n\}$ be a sequence of nonnegative numbers such that

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in real number such that

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$,

then $\lim_{n \rightarrow \infty} a_n = 0$.

2 Explicit viscosity iterative algorithm

In this section, a new iterative scheme for finding a common element of the set of solutions for a equilibrium problems and the set of common fixed point for an infinite family of mappings in Hilbert space, is introduced.

Theorem 2.1 *Let*

- C be a nonempty closed convex subset of a real Hilbert space H ,
- f be a ρ -contractive map on C ,
- $J = \{1, 2, \dots, k\}$ be a finite index set,
- For each $i \in J$, let G_i be a bifunction from $C \times C$ into R satisfying (A1) – (A4),
- A be a strongly positive linear bounded operator on H with coefficient $\varpi > 0$,
- $\{S_n\} : H \rightarrow H$ be a family of infinite nonexpansive mappings,
- $\bigcap_{i=1}^k F(W) \cap EP(G_i) \neq \emptyset$ where $F(W) = \bigcap_{j=1}^n F(S_j)$,
- $\{x_n\}$ be the sequence generated as following :

$$\left\{ \begin{array}{l} G_1(u_{n,1}, y) + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \\ G_2(u_{n,2}, y) + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \\ \vdots \\ G_k(u_{n,k}, y) + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \\ \theta_n = \frac{1}{k} \sum_{i=1}^k u_{n,i}, \\ y_n = \beta_n \gamma f(\theta_n) + (I - \beta_n A) \theta_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) W_n y_n, \end{array} \right.$$

where $\{W_n\}$ is a sequence defined by Eq.(1.2). Also, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $r_n \subset (0, \infty)$ and $0 < \gamma < \frac{\varpi}{\rho}$.

Suppose

- (C1): $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,
(C2): $\liminf_{n \rightarrow \infty} r_n > 0$, $\lim_{n \rightarrow \infty} (r_{n+1} - r_n) = 0$;
(C3): $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$,
(C4): for each $i = 1, 2, \dots, k$ $0 < \lambda_i \leq c < 1$.

Then

- (i) the sequence $\{x_n\}$ is bounded.
(ii) $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.
(iii) $\lim_{n \rightarrow \infty} \|W_n y_n - y_n\| = 0$.

Proof. From (C1), we may assume that $\beta_n \leq \|A\|^{-1}$ for all $n \geq 1$. By Lemma 1.5, we obtain $\|I - \beta_n A\| \leq 1 - \beta_n \varpi$. It is clear that $P_{\bigcap_{i=1}^k F(W) \cap EP(G_i)}(I - A - \gamma f)$ is a contraction of C into itself. Indeed, for all $x, y \in C$

$$\begin{aligned}
& \|P_{\bigcap_{i=1}^k F(W) \cap EP(G_i)}(x) - P_{\bigcap_{i=1}^k F(W) \cap EP(G_i)}(y)\| \\
& \leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\
& \leq \|I - A\| \|x - y\| + \gamma \|f(x) - f(y)\| \\
& \leq (1 - \varpi) \|x - y\| + \gamma \rho \|x - y\| \\
& = (1 - (\varpi - \gamma \rho)) \|x - y\|.
\end{aligned}$$

(i): Let $x^* \in \bigcap_{i=1}^k F(W) \cap EP(G_i)$. Since $u_{n,i} = T_{r_n,i} x_n$ and $x^* = T_{r_n,i} x^*$, we see for any $n \geq N$

$$\|u_{n,i} - x^*\| = \|T_{r_n,i} x_n - T_{r_n,i} x^*\| \leq \|x_n - x^*\|,$$

thus

$$\|\theta_n - x^*\| \leq \|x_n - x^*\|. \quad (2.1)$$

Since f is ρ -contraction, we have

$$\begin{aligned}
\|y_n - x^*\| &= \|\beta_n \gamma f(\theta_n) + (I - \beta_n A)\theta_n - x^*\| \\
&= \|\beta_n(\gamma f(\theta_n) - Ax^*) + (I - \beta_n A)(\theta_n - x^*)\| \\
&\leq \beta_n \|\gamma f(\theta_n) - Ax^*\| + \|I - \beta_n A\| \|\theta_n - x^*\| \\
&\leq \beta_n \gamma \|f(\theta_n) - f(x^*)\| \\
&\quad + \beta_n \|\gamma f(x^*) - Ax^*\| + (1 - \beta_n) \varpi \|x_n - x^*\|.
\end{aligned}$$

From which it follows that

$$\|y_n - x^*\| \leq (1 - \beta_n(\varpi - \gamma\rho))\|x_n - x^*\| + \beta_n \|\gamma f(x^*) - Ax^*\|. \quad (2.2)$$

In viwe of Eq. (2.1) and Eq.(2.2), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n x_n + (1 - \alpha_n)W_n y_n - x^*\| \\
&= \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(W_n y_n - x^*)\| \\
&\leq \alpha_n \|x_n - x^*\| + (1 - \alpha_n) \|y_n - x^*\| \\
&\leq \alpha_n \|x_n - x^*\| + (1 - \alpha_n) \{ (1 - \beta_n(\varpi - \gamma\rho)) \|x_n - x^*\| \\
&\quad + \beta_n \|\gamma f(x^*) - Ax^*\| \} \\
&= (1 - \alpha_n)(1 - \beta_n(\varpi - \gamma\rho)) \|x_n - x^*\| \\
&\quad + \beta_n(\varpi - \gamma\rho)(1 - \alpha_n) \frac{\|f(x^*) - Ax^*\|}{\varpi - \gamma\rho}.
\end{aligned}$$

It follows by induction that

$$\|x_{n+1} - x^*\| \leq \max \left\{ \|x_1 - x^*\|, \frac{\|f(x^*) - Ax^*\|}{\varpi - \gamma\rho} \right\}.$$

Therefore, the sequence $\{x_n\}$ is bounded and also $\{y_n\}, \{\theta_n\}$ are bounded.

(ii): Notic that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|(I - \beta_{n+1}A)(\theta_{n+1} - \theta_n) + (\beta_n - \beta_{n+1})A\theta_n \\
&\quad + \gamma\{\beta_{n+1}(f(\theta_{n+1}) - f(\theta_n)) + f(\theta_n)(\beta_{n+1} - \beta_n)\}\| \\
&\leq (1 - \beta_{n+1}\varpi)\|\theta_{n+1} - \theta_n\| + |\beta_n - \beta_{n+1}| \|A\theta_n\| \\
&\quad + \gamma\beta_{n+1}\rho\|\theta_{n+1} - \theta_n\| + \gamma|\beta_{n+1} - \beta_n| \|f(\theta_n)\|.
\end{aligned}$$

It follows that

$$\|y_{n+1} - y_n\| \leq (1 - \beta_{n+1}(\varpi - \gamma\rho))\|\theta_{n+1} - \theta_n\| + |\beta_{n+1} - \beta_n|M. \quad (2.3)$$

where $M = \text{Sup}_{n \geq 1} \{\|A\theta_n\| + \|\gamma(\theta_n)\|\}$.

Moreover, we have

$$G_i(u_{n+1,i}, u_{n,i}) + \frac{1}{r_{n+1}} \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} \rangle \geq 0, \quad (2.4)$$

$$1 \geq i \geq k. \quad (2.5)$$

and

$$G_i(u_{n,i}, u_{n+1,i}) + \frac{1}{r_n} \langle u_{n+1,i} - u_{n,i}, u_{n,i} - x_n \rangle \geq 0. \quad (2.6)$$

Combining Eq.(2.5) and Eq.(2.6), we obtain

$$\begin{aligned} 0 &\leq r_{n+1} \{G_i(u_{n+1,i}, u_{n,i}) + G_i(u_{n,i}, u_{n+1,i})\} \\ &\quad + \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n} (u_{n,i} - x_n) \rangle \\ &\leq \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n} (u_{n,i} - x_n) \rangle, \end{aligned}$$

from which it follows that

$$\langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n} (u_{n,i} - x_n) \rangle \leq 0 \quad (2.7)$$

which implies that

$$\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n} \|x_n - u_{n,i}\|. \quad (2.8)$$

Using the condition (C2) and noting that there exists $b > 0$ such that $r_n > b > 0$, we obtain

$$\|\theta_{n+1} - \theta_n\| \leq \frac{1}{k} \sum_{i=1}^k \|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n} M \quad (2.9)$$

and

$$\|\theta_{n+1} - \theta_n\| \leq \|x_{n+1} - x_n\| + \frac{\hat{M}}{b} |r_{n+1} - r_n|, \quad (2.10)$$

where $\hat{M} := \frac{1}{k} \sum_{i=1}^k \|x_n - u_{n,i}\| < \infty$.

Moreover, we note that

$$\begin{aligned} \|W_{n+1}y_n - W_ny_n\| &= \|\lambda_1 S_1 U_{n+1,2}y_n + (1 - \lambda_1)y_n \\ &\quad - (\lambda_1 S_1 U_{n,2}y_n + (1 - \lambda_1)y_n)\| \\ &\leq \lambda_1 \|U_{n+1,2}y_n - U_{n,2}y_n\| \\ &\leq \lambda_1 \|\lambda_2 S_2 U_{n+1,3}y_n + (1 - \lambda_2)y_n \\ &\quad - (\lambda_2 S_2 U_{n,3}y_n + (1 - \lambda_2)y_n)\| \\ &\leq \lambda_1 \lambda_2 \|U_{n+1,3}y_n - U_{n,3}y_n\| \\ &\quad \vdots \\ &\leq \left(\prod_{m=1}^n \lambda_m \right) \|U_{n+1,n+1}y_n - U_{n,n+1}y_n\| \\ &= \left(\prod_{m=1}^n \lambda_m \right) \|\lambda_{n+1} S_{n+1} U_{n+1,n+2}y_n \\ &\quad + (1 - \lambda_{n+1})y_n - y_n\| \\ &= \left(\prod_{m=1}^n \lambda_m \right) \|\lambda_{n+1} S_{n+1}y_n - \lambda_{n+1}y_n\| \end{aligned}$$

$$= \left(\prod_{m=1}^{n+1} \lambda_m \right) \|S_{n+1}y_n - y_n\| \leq \hat{M} \left(\prod_{m=1}^{n+1} \lambda_m \right) \quad (2.11)$$

where $\hat{M} := \sup_{n \geq 1} \{\|S_{n+1}y_n - y_n\|\}$.

Combining Eq.(2.3), Eq.(2.10) and Eq.(2.11), we obtain

$$\begin{aligned}
\|W_{n+1}y_{n+1} - W_n y_n\| &= \|W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n\| \\
&\leq \|y_{n+1} - y_n\| + \|W_{n+1}y_n - W_n y_n\| \\
&\leq \|\theta_{n+1}\theta_n\| + M|\beta_{n+1} - \beta_n| + \hat{M}\left(\prod_{m=1}^{n+1} \lambda_m\right) \\
&\leq \|x_{n+1} - x_n\| + \frac{\hat{M}}{b}|r_{n+1} - r_n| \\
&\quad + M|\beta_{n+1} - \beta_n| + \hat{M}\left(\prod_{m=1}^{n+1} \lambda_m\right).
\end{aligned}$$

We have

$$\limsup_{n \rightarrow \infty} (\|W_{n+1}y_{n+1} - W_n y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 1.6, we see that

$$\|W_n y_n - x_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (2.12)$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \alpha_n) \|W_n y_n - x_n\| = 0$$

(iii): We shall prove that $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Notic that

$$\begin{aligned}
\|u_{n,i} - x^*\|^2 &\leq \langle T_{r_{n,i}} x_n - T_{r_{n,i}} x^*, x_n - x^* \rangle \\
&= \frac{1}{2} \{ \|u_{n,i} - x^*\|^2 + \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2 \}
\end{aligned}$$

thus

$$\|u_{n,i} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2. \quad (2.13)$$

From Eq.(2.13), we get

$$\begin{aligned}\|\theta_n - x^*\| &= \left\| \sum_{i=1}^k \frac{1}{k} (u_{n,i} - x_n) \right\|^2 \\ &\leq \|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^k \|u_{n,i} - x_n\|^2.\end{aligned}\tag{2.14}$$

It follows from Eq.(2.14) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1 - \alpha_n)(W_n y_n - W_n x^*)\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \{ \|(I - \beta_n A)(\theta_n - x^*) \\ &\quad + \beta_n(\gamma f(\theta_n) - Ax^*)\|^2 \} \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \{ (1 - \beta_n \varpi) \|\theta_n - x^*\|^2 \\ &\quad + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \} \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|\theta_n - x^*\|^2 \\ &\quad + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \{ \|x_n - x^*\|^2 \\ &\quad - \frac{1}{k} \sum_{i=1}^k \|u_{n,i} - x_n\|^2 \} \\ &\quad + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\ &\leq \|x_n - x^*\|^2 - (1 - \alpha_n) \frac{1}{k} \sum_{i=1}^k \|u_{n,i} - x_n\|^2 \\ &\quad + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2.\end{aligned}$$

Thanks to the conditions of (C1)- (C4) and Eq.(2.13), we conclude that

$$\begin{aligned}(1 - \alpha_n) \frac{1}{k} \sum_{i=1}^k \|u_{n,i} - x_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\ &\leq \|x_{n+1} - x_n\| (\|x_{n+1} - x^*\| + \|x_n - x^*\|) \\ &\quad + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\ \lim_{n \rightarrow \infty} \|u_{n,i} - x_n\| &= 0, \text{ for each } i = 1, 2, \dots, k\end{aligned}$$

also

$$\lim_{n \rightarrow \infty} \|\theta_n - x_n\| = \lim_{n \rightarrow \infty} \|u_{n,i} - x_n\| = 0 \quad (2.15)$$

$$\|y_n - \theta_n\| = \beta_n \|\gamma f(\theta_n) - A\theta_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.16)$$

Moreover, we know that

$$\|y_n - x_n\| \leq \|x_n - \theta_n\| + \|\theta_n - y_n\|$$

$$\|W_n y_n - y_n\| \leq \|W_n y_n - x_n\| + \|x_n - \theta_n\| + \|\theta_n - y_n\|.$$

In viwe of Eq.(2.12), Eq.(2.15) and Eq.(2.16) , we can obtain

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0 \quad (2.17)$$

$$\lim_{n \rightarrow \infty} \|W_n y_n - y_n\| = 0. \quad (2.18)$$

□

Theorem 2.2 *Suppose all assumptions of Theorem 2.1 are holds. Then the sequence $\{x_n\}$ converge strongly to \tilde{x} , which solves the variational inequality*

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - x_n \rangle \leq 0, \quad \tilde{x} \in \bigcap_{i=1}^k F(W) \cap EP(G_i).$$

Equivalently, $P_{\bigcap_{i=1}^k F(W) \cap EP(G_i)}(I - A - \gamma f)(\tilde{x}) = \tilde{x}$.

Proof. We shall prove that

$$\limsup_{n \rightarrow \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle \leq 0,$$

where $x^* = P_{\bigcap_{i=1}^k F(W) \cap EP(G_i)} f(x^*)$.

We choose a subsequence $\{y_{n_p}\}$ of $\{y_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle = \lim_{p \rightarrow \infty} \langle (A - \gamma f)x^*, y_{n_p} - x^* \rangle, \quad (2.19)$$

since $\{y_{n_p}\}$ is bounded, there exists a subsequence of $\{y_{n_p}\}$, we denote it by $\{y_{n_p}\}$ such that $y_{n_p} \rightharpoonup q, q \in C$.

We shall show that $q \in \bigcap_{i=1}^k F(W) \cap EP(G_i)$. On the contrary, suppose that $q \notin F(W)$. By Opial's condition

$$\begin{aligned} \liminf_{p \rightarrow \infty} \|y_{n_p} - q\| &< \liminf_{p \rightarrow \infty} \|y_{n_p} - Wq\| \\ &\leq \liminf_{p \rightarrow \infty} \{\|y_{n_p} - Wy_{n_p}\| + \|Wy_{n_p} - Wq\|\} \\ &\leq \liminf_{p \rightarrow \infty} \{\|y_{n_p} - Wy_{n_p}\| + \|y_{n_p} - q\|\}. \end{aligned}$$

By virtue of Lemma 1.4 and noticing Eq.(2.18)

$$\begin{aligned} \lim_{p \rightarrow \infty} \|Wy_{n_p} - y_{n_p}\| &\leq \lim_{p \rightarrow \infty} \{\|Wy_{n_p} - W_{n_p}y_{n_p}\| + \|W_{n_p}y_{n_p} - y_{n_p}\|\} \\ &\leq \lim_{p \rightarrow \infty} \{Sup_{x \in C} \|Wx - W_{n_p}x\|\} \\ &\quad + \lim_{p \rightarrow \infty} \|W_{n_p}y_{n_p} - y_{n_p}\| = 0. \end{aligned}$$

It follows that

$$\liminf_{p \rightarrow \infty} \|y_{n_p} - q\| < \liminf_{p \rightarrow \infty} \|y_{n_p} - q\|.$$

This is a contradiction. Therefore, we have $q \in F(W)$. Also, we prove $q \in \bigcap_{i=1}^k EP(G_i)$.

For each $i \in J = \{1, 2, \dots, k\}$, since $G_i(u_{n_p}, y) + \frac{1}{r_{n_p}} \langle y, u_{n_p} - x_{n_p} \rangle \geq 0$,

from (A2), we see that

$$\begin{aligned} \frac{1}{r_{n_p}} \langle y - u_{n_p}, u_{n_p} - x_{n_p} \rangle &\geq G_i(u_{n_p}, y) + G_i(y, u_{n_p}) \\ &\quad + \frac{1}{r_{n_p}} \langle y - u_{n_p}, u_{n_p} - x_{n_p} \rangle \\ &\geq G_i(y, u_{n_p}), \end{aligned}$$

hence

$$\langle y - u_{n_p}, \frac{u_{n_p} - x_{n_p}}{r_{n_p}} \rangle \geq G_i(y, u_{n_p}), \text{ for all } y \in C.$$

Since $\frac{|u_{n_p} - x_{n_p}|}{r_{n_p}} \rightarrow 0$, $u_{n_p} \rightarrow q$, in viwe of (A4), we conclude

$$G_i(y, q) \leq 0, \text{ textbf forally } y \in C.$$

Let $0 < t \leq 1$, $y \in C$ and $y_t = ty + (1-t)q$. It is clear that $G_i(y_t, q) \leq 0$. From (A1)-(A4), we obtai

$$0 = G_i(y_t, y_t) \leq tG_i(y_t, y) + (1-t)G_i(y_t, q) \leq tG_i(y_t, y), G_i(y, q) \geq 0, \text{ textbf forally } y \in C.$$

Thus $q \in \bigcap_{i=1}^k EP(G_i)$.

From Eq.(2.19), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle &= \lim_{p \rightarrow \infty} \langle (A - \gamma f)x^*, y_{n_p} - x^* \rangle \\ &= \langle (A - \gamma f)x^*, x^* - q \rangle \leq 0. \end{aligned}$$

It follows from Eq.(2.17) and Eq.(2.19) that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (A - \gamma f)x^*, x^* - x_n \rangle &\leq \limsup_{n \rightarrow \infty} \langle (A - \gamma f)x^*, y_n - x_n \rangle \\ &\quad + \limsup_{n \rightarrow \infty} \langle (A - \gamma f)x^*, x^* - y_n \rangle \leq 0. \end{aligned}$$

Finally, we prove that $x_n \rightarrow q$ where $x^* = P_{\bigcap_{i=1}^k F(W) \cap EP(G_i)} f(x^*)$.

By virtue of Lemma 1.7

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(I - \beta_n A)(\theta_n - x^*) + \beta_n(\gamma f(\theta_n) - Ax^*)\|^2 \\
&\leq \|(I - \beta_n A)(\theta_n - x^*)\|^2 + 2\beta_n \langle \gamma f(\theta_n) - Ax^*, y_n - x^* \rangle \\
&\leq \|(I - \beta_n A)(\theta_n - x^*)\|^2 \\
&\quad + 2\beta_n \gamma \rho \|x_n - x^*\| \|y_n - x^*\| + 2\beta_n \langle \gamma f(\theta_n) - Ax^*, y_n - x^* \rangle \\
&\leq (1 - \beta_n \varpi)^2 \|x_n - x^*\|^2 + \beta_n \gamma \rho (\|x_n - x^*\|^2 + \|y_n - x^*\|^2) \\
&\quad + 2\beta_n \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle.
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \frac{(1 - \beta_n \varpi)^2 + \beta_n \gamma \rho}{1 - \beta_n \gamma \rho} \|x_n - x^*\|^2 \\
&\quad + \frac{2\beta_n}{1 - \beta_n \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \\
&\leq \left\{ 1 - \frac{2\beta_n(\varpi - \gamma \rho)}{1 - \beta_n \rho \gamma} \right\} \|x_n - x^*\|^2 \\
&\quad + \frac{2\beta_n(\varpi - \gamma \rho)}{1 - \beta_n \gamma \rho} \left\{ \frac{1}{\varpi - \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \right. \\
&\quad \left. + \frac{\beta_n \varpi^2}{2(\varpi - \gamma \rho)} L \right\},
\end{aligned}$$

where $L = \text{Sup}\{\|x_n - x^*\|\}$.

Also

$$\|x_{n+1} - x^*\|^2 \leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 \quad (2.20)$$

it follows from Eq.(2.20) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \left\{ 1 - (1 - \alpha_n) \frac{2\beta_n(\varpi - \gamma \rho)}{1 - \beta_n \rho \gamma} \right\} \|x_n - x^*\|^2 \\
&\quad + (1 - \alpha_n) \frac{2\beta_n(\varpi - \gamma \rho)}{1 - \beta_n \rho \gamma} \left\{ \frac{1}{\varpi - \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \right. \\
&\quad \left. + \frac{\beta_n \varpi^2}{2(\varpi - \gamma \rho)} L \right\}.
\end{aligned}$$

Let $\xi_n := (1 - \alpha_n) \frac{2\beta_n(\varpi - \gamma\rho)}{1 - \beta_n\rho\gamma}$ and

$$\varepsilon_n := \frac{1}{\varpi - \gamma\rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle + \frac{\beta_n \varpi^2}{2(\varpi - \gamma\rho)} L.$$

Therefore,

$$\|x_{n+1} - x^*\|^2 \leq (1 - \xi_n) \|x_n - x^*\|^2 + \xi_n \varepsilon_n. \quad (2.21)$$

Thanks to the condition (C1) and Eq.(2.21), we conclude that

$$\lim_{n \rightarrow \infty} \xi_n = 0, \quad \sum_{n=1}^{\infty} \xi_n = \infty.$$

From Lemma 1.10 we can obtain $x_n \rightarrow x^*$. \square

3 Numerical Example

In this section, we get one example is presented to guarantee the Theorem (2.2).

Example 3.1 Let $H = \mathbb{R}, C = [-1, 1]$ and $G_1(x, y) = -3x^2 + xy + 2y^2, G_2(x, y) = -4x^2 + xy + 3y^2$ and $G_3(x, y) = -9x^2 + xy + 8y^2$. Also, we consider $S_n = I, f(x) = \frac{x}{5}$ and $A = I$ be a strongly positive linear bounded operator with coefficient $\gamma = 1$. It is easy to check that A and f satisfy all conditions in Theorem 2.2. For each $r > 0$ and $x \in C$, there exists $z \in C$ such that, for any $y \in C$,

$$\begin{aligned} G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle &\geq 0 \\ \Leftrightarrow -3z^2 + zy + 2y^2 + \frac{1}{r} (y - z)(z - x) &\geq 0 \\ \Leftrightarrow 2ry^2 + ((r + 1)z - x)y - 3rz^2 - z^2 + zx &\geq 0 \end{aligned}$$

Set $G(y) = 2ry^2 + ((r+1)z - x)y - 3rz^2 - z^2 + zx$. Then $G(y)$ is a quadratic function of y with coefficients $a = 2r, b = (r+1)z - x$ and $c = -3rz^2 - z^2 + zx$. So

$$\begin{aligned}\Delta &= [(r+1)z - x]^2 - 8r(zx - z^2 - 3rz^2) \\ &= (r+1)^2z^2 - 2(r+1)xz + x^2 + 24r^2z^2 + 8rz^2 - 8rzx \\ &= x^2 - 2(5rz + z)x + (25r^2z^2 + 10rz^2 + z^2) \\ &= [(x - (5rz + z))]^2.\end{aligned}$$

Since $G(y) \geq 0$ for all $y \in C$, if and only if $\Delta = [(x - (5rz + z))]^2 \leq 0$. Therefore, $z = \frac{x}{5r+1}$, which yields $T_{r_n,1} = u_n^{(1)} = \frac{x_n}{5r_n+1}$.

By the same argument, for G_2 and G_3 , one can conclude $T_{r_n,2} = u_n^{(2)} \frac{x_n}{7r_n+1}$ and $T_{r_n,3} = u_n^{(3)} = \frac{x_n}{17r_n+1}$. Let $r_n = \frac{n}{n+1}$. Hence

$$\theta_n = \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)}}{3} = \frac{1}{3} \frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1} x_n.$$

Suppose that $\alpha_n = \frac{2n-1}{10n-9}, \beta_n = \frac{1}{n}$ and $\lambda_n = \varepsilon$, we have

$$\begin{aligned}W_1 &= U_{11} = \lambda_1 S_1 U_{12} + (1 - \lambda_1)I, \\ W_2 &= U_{21} = \lambda_1 S_1 U_{22} + (1 - \lambda_1)I \\ &= \lambda_1 S_1 \{ \lambda_2 S_2 U_{23} + (1 - \lambda_2)I \} + (1 - \lambda_1)I, \\ &= \lambda_1 \lambda_2 S_1 S_2 + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \\ W_3 &= U_{31} = \lambda_1 S_1 U_{32} + (1 - \lambda_1)I \\ &= \lambda_1 S_1 \{ \lambda_2 S_2 U_{33} + (1 - \lambda_2)I \} + (1 - \lambda_1)I, \\ &= \lambda_1 \lambda_2 S_1 S_2 U_{33} + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \\ &= \lambda_1 \lambda_2 S_1 S_2 \{ \lambda_3 S_3 U_{34} + (1 - \lambda_3)I \} + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \\ &= \lambda_1 \lambda_2 \lambda_3 S_1 S_2 S_3 + \lambda_1 \lambda_2 (1 - \lambda_3) S_1 S_2 + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I.\end{aligned}$$

By iteration this manner, we have

$$\begin{aligned}W_n &= U_{n1} = \lambda_1 \lambda_2 \cdots \lambda_n S_1 S_2 \cdots S_n + \lambda_1 \lambda_2 \cdots \lambda_{n-1} (1 - \lambda_n) S_1 S_2 \cdots S_{n-1} \\ &\quad + \lambda_1 \lambda_2 \cdots \lambda_{n-2} (1 - \lambda_{n-1}) S_1 S_2 \cdots S_{n-2} + \cdots + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I.\end{aligned}$$

Let $T_n = I, \lambda_n = \varepsilon$, we obtain

$$W_n = [\varepsilon^n + \varepsilon^{n-1}(1 - \varepsilon) + \cdots + \varepsilon(1 - \varepsilon) + (1 - \varepsilon)]I = I.$$

Hence

$$y_n = \left(\frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1} \right) \left(\frac{15n - 14}{15n} \right) x_n.$$

We have the following algorithm for the sequence $\{x_n\}$

$$x_{n+1} = \frac{2n - 1}{10n - 9} x_n + \frac{8n - 8}{10n - 9} y_n.$$

Choose $x_1 = 1$. By using MATLAB software, we obtain the following table and figure of the result.

n	x_n	n	x_n	n	x_n
1	1.0	11	0.003385691332	21	0.000002071711754
2	1.270565302	12	0.001637456001	22	0.0000009778618741
3	0.7706281483	13	0.0007885980277	23	0.0000004609899888
4	0.4251825949	14	0.0003784282172	24	0.0000002170787103
5	0.2242603285	15	0.0001810390789	25	0.0000001021162838
6	0.1151384546	16	0.00008637699279	26	0.00000004799105366
7	0.05805496956	17	0.00004111536174	27	0.00000002253432275
8	0.02889456779	18	0.00001953029496	28	0.00000001057245007
9	0.01424086637	19	0.000009260000453	29	0.000000004956540717
10	0.006965124578	20	0.000004383229046	30	0.000000002322073954

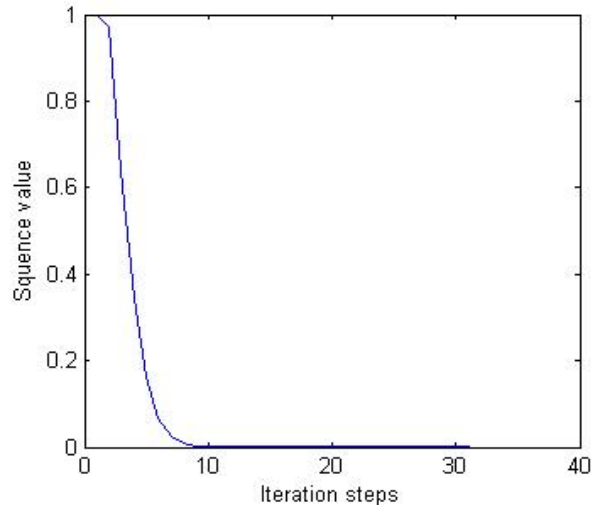


Fig. 1. The graph of $\{x_n\}$ with initial value $x_1 = 1$.

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