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A New Technique for Solving Fredholm Integro-Differential Equations Using the Reproducing Kernel Method

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Abstract

This paper is concerned with a technique for solving Fredholm integro-differential equations in the reproducing kernel Hilbert space. In contrast with the conventional reproducing kernel method, the Gram-Schmidt process is omitted here and satisfactory results are obtained. The analytical solution is represented in the form of series. An iterative method is given to obtain the approximate solution. The convergence analysis is established theoretically. The applicability of the iterative method is demonstrated by testing some various examples.

Key words: Reproducing kernel method,Integro-differential equations,Gram-Schmidt orthogonalization process.
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1 Introduction

Recently, the reproducing kernel method (RKM) has been a promising method which applied more and more for solving various problems such as ordinary differential equations, partial differential equations, differential-difference equations, integral equations, and so on (see e.g. [1]-[18] and references there in). Among many literatures related to RKM for solving various problems and even among a bunch of extensive works related to RKM for solving integro - differential and integral equations, we just mention some more interesting problems. An approximate solution of the Fredholm integral equation of the first kind in the reproducing kernel space was presented by Du and Cui [5,6], solution of a system of the linear Volterra integral equations was discussed by Yang et al. [18], solvability of a class of Volterra integral equations with weakly singular kernel using RKM was investigated in [2,3,11], Geng [9] explained how to solve a Fredholm integral equation of the third kind in the reproducing kernel space, and Ketabchi et al. [12] obtained some error estimates for solving Volterra integral equations using RKM.

In [1] and some other places, a general technique for solving integro-differential equations was discussed in the reproducing kernel space. This general technique is based on the Gram-Schmidt (GS) orthogonalization process. In this study, we aim to explain how to construct a reproducing kernel method without using this process. For this purpose, we consider the following nonlinear Fredholm integro - differential equation

$$u'(x) = F(x, u(x)) + Su(x) = T(x, u(x), Su(x)), \tag{1.1}$$

where

$$Su(x) = \int_a^b k(x,s)\mathcal{G}(u(s))ds,$$

subject to the initial condition $u(a) = \alpha$ in which functions $\mathcal{G}$ and k and the nonlinear operator $\mathcal{G}$ are considered such that Eq.(1.1) has a unique solution. Furthermore, we need to assume that F,T are continuous.

The rest of the paper is organized as follows. In the next Section, some preliminaries are represented. The method implementation is discussed in Section 3. Section 4 is devoted to convergence analysis of the method. For confirming the theoretical results, some numerical examples are provided in Section 5. The paper will be closed by a brief conclusion in the last Section.

2 Preliminaries

In this section, we follow the recent work by Cui et al. [4] and represent some useful materials.

Definition 2.1 *Let H be a Hilbert space of functions $f : \Omega \to \mathbb{R}$. Denote by $< .,. >$ the inner product and let $\|.\| = \sqrt{< .,. >}$ be the induced norm in H. The function $R : \Omega \times \Omega \to \mathbb{R}$ is called a reproducing kernel of H if the followings are satisfied*

(1) $R_y(x) = R(x, y) \in H, \forall y \in \Omega,$
(2) $f(y) =< f(x), R_y(x) >, \forall f \in H,\ \forall y \in \Omega.$

Definition 2.2 *A Hilbert space H of functions on a set Ω is called a reproducing kernel Hilbert space if there exists a reproducing kernel R of H.*

Remark 2.1 *The existence of the reproducing kernel of a Hilbert space is due to the Riesz Representation Theorem. It is known that the reproducing kernel of a Hilbert space is unique.*

Theorem 2.1 *[16] The reproducing kernel R of reproducing kernel Hilbert space H is positive definite.*

Definition 2.3 *The function space $W_2[a, b]$ is defined as follows*

$$W_2[a, b] = \{u|u, u' \in AC[a, b], u, u', u^{(2)} \in L^2[a, b], u(a) = 0\}.$$

AC is Absolute Continuous.

The inner product and norm in $W_2[a,b]$ are defined respectively by

$$< u, v >_{W_2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u^{(2)}(x)v^{(2)}(x)dx,$$

$$\forall u, v \in W_2[a,b],$$

and

$$\|u\|_{W_2} = \sqrt{< u, u >_{W_2}}, \qquad \forall u \in W_2[a,b].$$

The function space $W_2[a,b]$ is a reproducing kernel space and its reproducing kernel R_2 has the following reproducing property

$$u(.) = < u(x), R_2(x,.) >_{W_2}, \qquad \forall u \in W_2[a,b].$$

The function space $W_2[a,b]$ is a reproducing kernel space and its reproducing kernel is [1]

$$R_2(x,y) = \begin{cases} \frac{1}{6}(x-a)(2a^2 - x^2 + 3y(2+x) - a(6+3y+x)) & x \leq y, \\ \frac{1}{6}(y-a)(2a^2 - y^2 + 3x(2+y) - a(6+3x+y)) & x > y. \end{cases}$$

Similarity the function space is a reproducing kernel space and its reproducing kernel is [5]

$$W_1[a,b] = \{u | u \in AC[a,b], u' \in L^2[a,b], u(a) = 0\}$$

$$R_1(x,y) = \begin{cases} 1 - a + x, & x \leq y, \\ 1 - a + y, & x > y. \end{cases}$$

3 The method implementation

We rewrite Eq. (1.1) as follows

$$Lu(x) = u'(x) = T(x, u(x), Su(x)) = F(x, u(x)) + \int_a^b k(x,s)\mathcal{G}(u(s))ds,$$

where $L : W_2[a,b] \to W_1[a,b]$ is an invertible bounded linear operator [1], $\mathcal{G}$ is a nonlinear and continuous operator , and F, is an arbitrary continuous function in $W_1[a,b]$. $W_2[a,b]$ is a reproducing kernel space defined according to the highest derivatives involved in (1.1).
We choose a countable set of points $\{x_i\}_{i=1}^{\infty}$ in the interval $[a,b]$, and define

$$\phi_i(x) = R_1(x, x_i), \qquad \psi_i(x) = L^*\phi_i(x),$$

where L^* is the adjoint operator of L. Obviously,

$$\psi_i(x) = L^*\phi_i(x) = < L^*\phi_i(x), R_2(x,y) >_{W_2} = < \phi_i(x), L_y R_2(x,y) >_{W_1}$$

$$= L_y R_2(x,y)|_{y=x_i}$$

where L_y indicates that the operator L applies to the function of y.

Theorem 3.1 *Let $\{x_i\}_{i=1}^{\infty}$ be dense in the interval $[a,b]$. If Eq. (1.1) has a unique solution, then it can be represented as*

$$u(x) = \sum_{j=1}^{\infty} a_j \psi_j(x), \tag{3.1}$$

where the coefficients a_j are determined by solving the following semi-infinite system of linear equations

$$Ba = T, \tag{3.2}$$

in which

$$B = [L\psi_j(x_i)], \qquad i, j = 1, 2, \ldots, \qquad a = [a_1, a_2, \ldots]^T,$$

and

$$T = [T(x_1, u(x_1), Su(x_1)), T(x_2, u(x_2), Su(x_2)), \ldots]^T.$$

Proof. Since $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a,b]$, then $\psi_j(x)$ is a complete system in $W_2[a,b]$, see e.g.[4]. So the analytical solution can be represented as Eq. (3.1). Since

$$< \psi_i(x), \psi_j(x) >_{w_2} = < L^*\phi_i(x), \psi_j(x) >_{w_2} = < \phi_i(x), L\psi_j(x) >_{w_1}$$

$$= L\psi_j(x)|_{x=x_i}$$

and

$$< u(x), \psi_j(x) >_{w_2} = < u(x), L^*\phi_j(x) >_{w_2} = < Lu(x), \phi_j(x) >_{w_1}$$

$$= T(x_j, u(x_j), Su(x_j)),$$

according to the best approximation in Hilbert spaces [16], the coefficients a_j are determined by (3.2). □

The approximate solution of the problem is the m-term intercept of the analytical solution which can be determined by solving a $m \times m$ system of linear equations. We need to construct an iterative method for solving (3.2). For this purpose, we choose the number of points m, the number of iterations n and put the initial function $u_{a,m}(x) = 0$. Then, the approximate solution of Eq. (1.1) is defined by

$$\sum_{j=1}^{m} a_j^n L(\psi_j(x_i)) = T(x_i, u_{n-1,m}(x_i), Su_{n-1,m}(x_i)). \tag{3.3}$$

Remark 3.1 *There exists a unique solution for equations (3.3) due to the strictly positive definiteness property of the reproducing kernel.*

Theorem 3.2 *[12] The approximate solution $u_{n,m}$ and its derivative $u'_{n,m}$ are both uniformly convergent.*

The results of this section can be summarized in the following algorithm. ***ALGORITHM1***

(1) Choose m collocation points in the interval $[a, b]$.
(2) Set $B = [L\psi_j(x_i)], \quad i, j = 1, 2, \ldots, m.$
(3) Choose the number of iterations n.
(4) Set $i = 0$.
(5) Set the initial function $u_{a,m}(x) = 0$.

(6) Set $i = i + 1$.
(7) Set $T = [T(x_j, u_{i-1,m}(x_j), Su_{i-1,m}(x_j))]^T, \; j = 1, \ldots, m$.
(8) Solve $Ba = T$.
(9) Set $u_{i,m}(x) = \sum_{j=1}^{m} a_j^i \psi_j(x)$.
(10) If $i < n$, then go to step 6, else stop .

The conventional reproducing kernel method which used the GS orthogonalization process is represented in the following algorithm [1].

ALGORITHM2

(1) Choose m collocation points in the domain set $[a, b]$.
(2) Set $\phi_i(x) = R_1(x, x_i), \qquad i = 1, \ldots, m$.
(3) Set $\psi_i(x) = L_y R_2(x, x_i)$
(4) Set $\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \quad i = 1, \ldots, m$, ($\beta_{ik}$ which obtained by the GS process).
(5) Choose an initial function $u_0(x)$.
(6) Set $n = 1$.
(7) Set $B_n = \sum_{l=1}^{n} \beta_{nl} T(x_l, u_{n-1}(x_l), Su_{n-1}(x_l))$.
(8) Set $u_n(x) = \sum_{j=1}^{n} B_j \bar{\psi}_j(x)$.
(9) If $n < m$, then set $n = n + 1$ and go to step 7, else stop.

Remark 3.2 *In comparison with Algorithm 2, Algorithm 1 needs not to use the GS orthogonalization process but in step 8 of it, we must to solve a linear system. The coefficient matrix of this system is positive definite because of the positive definiteness of the kernel. Therefore, it needs to decompose matrix B once using the QR decomposition and to solve a triangular system in step 8.*

4 Convergence analysis

In this section, we show that the approximate solution $u_{n,m}$ is converged to the analytical solution u uniformly. At first, the following lemma is given.

Lemma 4.1 *For a positive constant M, $A = \{u| \quad \|u\|_{w_2} \leq M\}$ is a compact set in the space $C[a,b]$ provided that*

$$\|u'\|_{w_2} \leq c,$$

where c is a constant.

Proof. It is enough to show that A is a bounded and equicontinuous set [16]. Since

$$\|R_2(x,y)\|^2_{w_2} =< R_2(x,y), R_2(x,y) >_{w_2}= R_2(x,x) < c_0,$$

where c_0 is a positive constant, there exists a constant c_1 such that

$$|u(x)| = | < u(y), R_2(x,y) >_{w_2} | \leq \|u(y)\|_{w_2}\|R_2(x,y)\|_{w_2}$$

$$\leq c_1\|u(y)\|_{w_2}.$$

For any $u \in A$, we have

$$|u(x)| \leq c_1 M$$

Hence A is a bounded set in the space $C[a,b]$. On the other hand,

$$|u'(x)| = | < u(y), \frac{\partial R_2(x,y)}{\partial x} >_{w_2} | \leq \|u(y)\|_{w_2}\|\frac{\partial R_2(x,y)}{\partial x}\|_{w_2}$$

$$\leq c_2\|u(y)\|_{w_2} \leq c_2 M$$

Then for any $u \in A$ and $\epsilon > 0$, we have

$$|u(x+h) - u(x)| \leq |u'(\eta)||h| \leq c_2 M|h|$$

where $\eta \in [x, x+h]$. So, there exists $\delta = \frac{\epsilon}{c_2 M}$ such that for $|h| < \delta$, we get

$$|u(x+h) - u(x)| < \epsilon$$

Hence A is an equicontinuous set in the space $C[a,b]$.■

Theorem 4.1 *If L is an invertible bounded linear operator and $T(x, u(x), Su(x))$ is a nonlinear bounded operator, it can be deduced that $\{u_{n,m}(x)\}_{n=1}^{\infty}$ is the bounded sequence of functions in $w_2[a,b]$.*

proof. We can write

$$\begin{aligned}
\|u_{n,m}(x)\|_{w_2}^2 &=< u_{n,m}(x), u_{n,m}(x) >_{w_2} \\
&=< \textstyle\sum_{j=1}^m a_j\psi_j(x), \sum_{l=1}^m a_l\psi_l(x) >_{w_2} \\
&= \textstyle\sum_{j=1}^m a_j < \psi_j(x), \sum_{l=1}^m a_l\psi_l(x) >_{w_2} \\
&= \textstyle\sum_{j=1}^m a_j < \phi_j(x), \sum_{l=1}^m a_l L\psi_l(x) >_{w_1} \\
&= \textstyle\sum_{j=1}^m a_j (\sum_{l=1}^m a_l L\psi_l(x_j)) \\
&= a^T B a,
\end{aligned}$$

where

$$a = [a_j], \qquad j = 1, 2, \ldots, m.$$

Now, since

$$B = [L\psi_j(x_i)], \qquad i, j = 1, 2, \ldots, m,$$

$$a = B^{-1}T$$

the assumptions imply that

$$\|u_{n,m}(x)\|_{w_2} \leq M,$$

where M is a constant. $\square$

Theorem 4.2 *Assume that $\{x_i\}_{i=1}^\infty$ is dense in $[a, b]$ and the assumptions of Theorem (4.1) and Lemma (4.1) hold. Then the approximate solution $u_{n,m}$ is converged to the analytical solution u.*

proof. For $j = 1, 2, \ldots, m$ and $n = 1, 2, \ldots$, we have

$$Lu_{n,m}(x_j) = T(x_j, u_{n-1,m}(x_j), Su_{n-1,m}(x_j)).$$

According to Lemma (4.1), there exists a convergent subsequence $\{u_{n_l,m}(x)\}_{l=1}^\infty$ of $\{u_{n,m}(x)\}_{n=1}^\infty$ such that $u_{n_l,m}(x) \to u_{n,m}(x)$, uniformly as $l \to \infty$, $m \to \infty$. Then for $j = 1, 2, \ldots, m$ and $n =$

$1, 2, \ldots$, we derive

$$Lu_{n_l,m}(x_j) = T(x_j, u_{n_l-1,m}(x_j), Su_{n_l-1,m}(x_j)). \tag{4.1}$$

Since the operators L and T are both continuous (according to the structure of L and assumption on T), after taking limit from both sides of (4.1), it can be inferred that u is the analytical solution of Eq. (1.1). So $u_{n_l,m}(x)$ is the approximate solution of Eq. (1.1). $\square$

5 Numerical results

In this section, we compare results of both Algorithms in solving four various problems using the following norms

$$\begin{aligned}
\|u - u_{n,m}\|_\infty &\simeq E_{n,m} = \max_{1\leq i\leq m} |u(x_i) - u_{n,m}(x_i)|,\\
\|u' - u'_{n,m}\|_\infty &\simeq E'_{n,m} = \max_{1\leq i\leq m} |u'(x_i) - u'_{n,m}(x_i)|,\\
\|u - u_m\|_\infty &\simeq E_m \;\;\, = \max_{1\leq i\leq m} |u(x_i) - u_m(x_i)|
\end{aligned}$$

where $u_{n,m}$ and u_m are approximate solutions obtained by Algorithms 1 and 2, respectively and u is the exact solution and $u'_{n,m}$ is the derivative approximate solution obtained by Algorithm 1 ,u' is the derivative exact solution. The results of Table 1 and Table 2 (for $n = 5$) confirm the superiority of Algorithm 1.

Example 5.1 *If $F(x, u(x)) = 1 - \frac{1}{3}x$, $Su(x) = \int_0^1 xu^2(s)ds$ and $u(0) = 0$, then the Fredholm integro-differential equation* (1.1) *has the following exact solution $u(x) = x$.*

Example 5.2 *If $F(x, u(x)) = -u(x) + \frac{1}{2}(exp(-2) - 1)$, $Su(x) = \int_0^1 u^2(s)ds$ and $u(0) = 1$, then the Fredholm integro-differential equation* (1.1) *has the following exact solution $u(x) = \exp(-x)$.*

Example 5.3 *If $F(x, u(x)) = \cos(x) - \frac{1}{4}x$, $Su(x) = \frac{-1}{4}\int_0^{\frac{\pi}{2}} xsu(s)ds$ and $u(0) = 0$,then the Fredholm integro - differential equation* (1.1) *has the following exact solution $u(x) = \sin(x)$.*

Table 1
Results of Algorithms 1 and 2

	Example 1		Example 2	
m	E_m	$E_{5,m}$	E_m	$E_{5,m}$
2	1.00000	$5.953285E-4$	0.63212	0.05508
4	0.09259	$5.953282E-4$	0.59122	0.00591
8	0.01170	$5.953279E-4$	0.57410	0.00093
16	0.00262	$5.953267E-4$	0.57339	0.00005
32	0.00060	$5.953252E-4$	0.57128	0.00001

	Example 3		Example 4	
m	E_m	$E_{5,m}$	E_m	$E_{5,m}$
2	1.00000	0.16437	0.54308	0.04361
4	0.56677	0.01709	0.25554	0.00494
8	0.06062	0.00302	0.05140	0.00090
16	0.02109	0.00050	0.01096	0.00019
32	0.01776	0.00024	0.00176	0.00004

Example 5.4 *If* $F(x,u(x)) = \sinh(x) + \frac{1}{8}(1-exp(-1))x$, $Su(x) = \frac{-1}{8}\int_0^1 xsu(s)ds$ *and* $u(0) = 1$ *,then the Fredholm integro - differential equation* (1.1) *has the following exact solution* $u(x) = \cosh(x)$.

Table 2
Results of Algorithms 1 for $E'_{5,m}$

m	Example 1	Example 2	Example 3	Example 4
2	$1.1906570E-4$	0.027062	0.06395	0.001815
4	$1.1906556E-4$	0.002455	0.00797	0.000156
8	$1.1906549E-4$	0.000561	0.00189	0.000027
16	$1.1906528E-4$	0.000067	0.00081	0.000006
32	$1.1906516E-4$	0.000042	0.00058	0.000001

6 Conclusion

In this work, we proposed an iterative algorithm for solving nonlinear Fredholm integro - differential equations on the basis of the reproducing kernel Hilbert space without using the Gram-Schmidt orthogonalization process. The results of some numerical examples show that the present method could be an accurate and reliable analytical-numerical technique. Examples presented here belong to different categories such as linear or nonlinear problem .Nevertheless, our results only apply to the given examples; this, of course, does not mean that it holds in general. The advantage of the approach is that the method can be easily implemented. It seems that the method can be also applied for solving other nonlinear integro-differential equations.

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