

# Elliptic Function solutions of (2+1)-Dimensional Breaking Soliton Equation by Sinh-Cosh Method and Sinh-Gordon Expansion Method

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## Abstract

In this paper, based on sinh-cosh method and sinh-Gordon expansion method, families of solutions of (2+1)-dimensional breaking soliton equation are obtained. These solutions include Jacobi elliptic function solution, soliton solution, trigonometric function solution.

*Key words:* sinh-cosh method, soliton, Jacobi elliptic function, sinh-Gordon expansion method.

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## 1 Introduction

There exist many methods for obtaining solutions of the (2+1)-Dimensional breaking soliton equation, such as the Generalized Jacobi elliptic function method [2],  $(G'/G)$  Expansion method [3] and so on.

In this paper, by using the sinh-cosh method [1] and sinh-Gordon expansion method [4,5], we construct elliptic function solutions in the (2+1)-dimensional breaking soliton equation.

$$u_t - bu_{xxy} + 4b(uv)_x = 0, \quad (1.1)$$

$$v_x - u_y = 0, \quad (1.2)$$

Where  $b$  is an arbitrary constant, the system (1)-(2) was used to describes the (2+1)-dimensional interaction of Riemann was propagated along the  $y$ -axis with long wave propagated along the  $x$ -axis and it seems to have been investigated extensively where over lapping solutions have been derived.

## 2 Methods

Consider a given (2+1)-dimensional breaking soliton equation with independent variable  $x = (t, x_1, x_2, \dots)$  and dependent variables  $u(x)$ . The following formal solution of the given (2+1)-dimensional breaking soliton equation will be sought by the following ansatz

$$u(x) = A_0 + \sum_{i=1}^n \cosh^{i-1}(w) [A_i \sinh(w) + B_i \cosh(w)], \quad (2.1)$$

Where  $n$  is an integer which is determined by balancing the highest order derivative term with the highest order nonlinear term in the given(1)-(2) [5], and  $A_0 = A_0(x), \dots, A_n = A_n(x), B_1 = B_1(x), \dots, B_n = B_n(x), w = w(\mu), \mu = \alpha x + p + q$  Are all differentiable function.

satisfies  $\omega$

$$\left(\frac{dw}{dx}\right)^2 = \sinh^2(w(\mu)) + c, \quad (2.2)$$

Or in another form

$$\frac{d^2w}{dx^2} = \sinh(w) \cosh(w), \quad (2.3)$$

Where  $c = 1 - m^2$  and  $m$  is the modulus of Jacobi elliptic function. Equation (2.2) has the following solution:

$$\sinh(w) = cs(\mu, m) = \frac{cn(\mu, m)}{sn(\mu, m)}, \quad (2.4)$$

$$\cosh(w) = ns(\mu, m) = \frac{1}{sn(\mu, m)}, \quad (2.5)$$

Where  $sn(\mu, m)$ ,  $cn(\mu, m)$  are jacobian elliptic sine function and the jacobian elliptic cosine function respectively. we can also seek (2+1)-dimensional breaking soliton equation s solution in the up form where  $w = a(\xi)$ ,  $\xi = k(x + \alpha y - \beta t)$  where  $\xi$  a real parameter and  $k, \alpha, \beta$  are constant.

### 3 the application of methods

#### 3.1 the application of sinh-cosh method

Inother to solve (1) and (2) by using our method , we first reduce (1) and (2) to a differential equations .we make transformations

$$u(x, y, t) = u(\mu), v(x, yt) = v(\mu), \quad (3.1)$$

$$\mu = \alpha x + p + q, \quad (3.2)$$

Where  $\alpha$  is a nonzero constant and  $p$  is the function of,  $q$  is a function  $t$ . The substitutions of (8) and (9) into (1) and (2) yields

$$q'(t)u' - b\alpha^2 p' \beta(y)u''' + 4b\alpha u'v + 4b\alpha wv' = 0, \quad (3.3)$$

$$\alpha v' - p'(y)u' = 0, \quad (3.4)$$

And integrating yields, (10) and (11)

$$q'(t) - b\alpha^2 p' \beta(y)u'' + 4b\alpha wv = 0, \quad (3.5)$$

$$\alpha v - p'(y)u = 0, \quad (3.6)$$

The substitutions of  $v = \frac{p'(y)}{\alpha}u$  into (12) yields

$$q'(t)u - b\alpha^2 p' \beta(y)u'' + 4bp'u^2 = 0. \quad (3.7)$$

Balancing  $u^2$  with  $u''$  then gives  $n = 2$ .

According to method we assume that (14) has the solution

$$u(x) = A_0 + A_1 \sinh(w) + B_1 \cosh(w) + A_2 \sinh(w) \cosh(w) + B_2 \cosh^2(w), \quad (3.8)$$

Substituting (15) into (14) along with (4) and (5), yields a differential equation about setting the coefficients of  $\sinh^i(w) \cosh^j(w) (\sinh^2(w) + c)^{\frac{k}{2}}$ ,  $i = 1, 2, \dots; j = 0, 1; k = 0, 1$ .

$\sinh^i(w) \cosh^j(w) (\sinh^2(w) + c)^{\frac{k}{2}}$ ,  $i = 1, 2, \dots; j = 0, 1; k = 0, 1$  to zero, we

get the overdetermined equations:

$$\begin{aligned}
& q'(t)A_0 + q'(t)B_2 + 2b\alpha^2 p'(y)B_2c + 4bp'(y)A_0^2 + 4bp'(y)B_1^2 + 4bp'(y)B_2^2 \\
& + 8bp'(y)A_0B_2 = 0, \\
& q'(t)A_1 - b\alpha^2 p'(y)A_1 - b\alpha^2 p'(y)A_1c + 8bp'(y)A_0A_1 + 8bp'(y)A_1B_2 \\
& + 8bp'(y)B_1A_2 = 0, \\
& q'(t)A_1 - b\alpha^2 p'(y)B_1c + 8bp'(y)A_0B_1 \\
& + 8bp'(y)B_1B_2 = 0, \\
& q'(t)A_2 - 4\alpha^2 bp'(y)A_2c - b\alpha^2 p'(y)A_2 + 8bp'(y)A_1B_1 + 8bp'(y)A_2B_2 \\
& + 8bp'(y)A_0A_2 = 0, \\
& q'(t)B_2 - 4\alpha^2 bp'(y)A_2c - 4\alpha^2 bp'(y)B_2c + 4bp'(y)A_1^2 + 4bp'(y)A_2^2 \\
& + 8bp'(y)B_2^2 \\
& + 8bp'(y)A_0B_2 + 4bp'(y)B_1^2 = 0, \\
& 8bp'(y)B_1A_2 - 2b\alpha^2 p'(y)A_1 + 8bp'(y)A_1B_2 = 0, \\
& 8bp'(y)A_1A_2 + 8bp'(y)B_1B_2 - 2b\alpha^2 p'(y)B_1 = 0, \\
& -6b\alpha^2 p'(y)A_2 + 8bp'(y)A_2B_2 = 0, \\
& -6b\alpha^2 p'(y)B_2 + 4bp'(y)A_2^2 + 4bp'(y)B_2^2 = 0.
\end{aligned}$$

Solving equations with Maple, we derive the solutions of the partial differential equations.

$$\begin{aligned}
A_0 &= \frac{1}{2}\alpha^2 \sqrt{\frac{1}{16} - c + c^2} - \frac{5}{8}\alpha^2 + \frac{1}{2}\alpha^2 c, A_1 = 0, B_1 = 0, \\
A_2 &= -\frac{3}{4}\alpha^2, B_2 = \frac{3}{4}\alpha^2, p = \frac{1}{\alpha^2}y, q = \left[ -4b\sqrt{\frac{1}{16} - c + c^2} \right] t
\end{aligned} \tag{3.9}$$

We have obtained solutions of (12) and (13) if  $v = \frac{1}{\alpha}p'(y)$ , these solutions are

$$\left\{ \begin{array}{l} u_{11} = \frac{1}{2}\alpha^2\sqrt{\frac{1}{16} - c + c^2} - \frac{5}{8}\alpha^2 + \frac{1}{2}\alpha^2c \\ \quad - \frac{3}{4}\alpha^2cs \left( \alpha x + \frac{1}{\alpha^2}y + \left[ -4b\sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\ \quad + ns \left( \alpha x + \frac{1}{\alpha^2}y + \left[ -4b\sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\ \quad + \frac{3}{4}\alpha^2ns^2 \left( \alpha x + \frac{1}{\alpha^2}y + \left[ -4b\sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\ v_{11} = \frac{1}{2\alpha}\sqrt{\frac{1}{16} - c + c^2} - \frac{5}{8\alpha} + \frac{1}{2\alpha}c \\ \quad - \frac{3}{4\alpha}cs \left( \alpha x + \frac{1}{\alpha^2}y + \left[ -4b\sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\ \quad + ns \left( \alpha x + \frac{1}{\alpha^2}y + \left[ -4b\sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \\ \quad + \frac{3}{4\alpha}ns^2 \left( \alpha x + \frac{1}{\alpha^2}y + \left[ -4b\sqrt{\frac{1}{16} - c + c^2} \right] t, m \right) \end{array} \right. \quad (3.10)$$

When  $m \rightarrow 1$ ,  $cs(\mu, m) \rightarrow csch(\mu)$  and  $ns(\mu, m) \rightarrow coth(\mu)$ ,  $c \rightarrow 0$  so obtain the following soliton solutions of (1) and(2). (figure 1)

$$\left\{ \begin{array}{l} u_{12} = -\frac{1}{2}\alpha^2 - \frac{3}{4}\alpha^2csch \left( \alpha x + \frac{1}{\alpha^2}y - bt \right) coth \left( \alpha x + \frac{1}{\alpha^2}y - bt \right) \\ \quad + \frac{3}{4}\alpha^2coth^2 \left( \alpha x + \frac{1}{\alpha^2}y - bt \right) \\ v_{12} = -\frac{1}{2\alpha} - \frac{3}{4\alpha}csch \left( \alpha x + \frac{1}{\alpha^2}y - bt \right) coth \left( \alpha x + \frac{1}{\alpha^2}y - bt \right) \\ \quad + \frac{3}{4}\alpha^2coth^2 \left( \alpha x + \frac{1}{\alpha^2}y - bt \right) \end{array} \right. \quad (3.11)$$

When  $m \rightarrow 0$ ,  $cs(\mu, m) \rightarrow coth(\mu)$  and  $ns(\mu, m) \rightarrow csc(\mu)$ ,  $c \rightarrow 1$  so obtain the following trigonometric function solutions of (1) and (2). (figure

2)

$$\begin{cases} u_{13} = -\frac{3}{4}\alpha^2 \coth\left(\alpha x + \frac{1}{\alpha^2}y - bt\right) \csc\left(\alpha x + \frac{1}{\alpha^2}y - bt\right) \\ \quad + \frac{3}{4}\alpha^2 \csc^2\left(\alpha x + \frac{1}{\alpha^2}y - bt\right), \\ v_{13} = -\frac{3}{4}\alpha \coth\left(\alpha x + \frac{1}{\alpha^2}y - bt\right) \csc\left(\alpha x + \frac{1}{\alpha^2}y - bt\right) \\ \quad + \frac{3}{4\alpha} \csc^2\left(\alpha x + \frac{1}{\alpha^2}y - bt\right) \end{cases} \quad (3.12)$$

### 3.2 the application of sinh-Gordon expansion method

In other to solve (1) and (2) by using our method, we first reduce (1) and (2) to differential equations. we make transformations

$$u(x, y, t) = u(\xi), v(x, y, t) = v(\xi) \quad (3.13)$$

$$\xi = k(x + \alpha y - \beta t) \quad (3.14)$$

Where  $\xi$  is real parameters and  $k, \alpha, \beta$  are constant. The substitutions of (20) and (21) into (1) and (2) yields

$$-k\beta u' - bk^3\alpha u''' + 4bku'v + 4bkuv' = 0, \quad (3.15)$$

$$kv' - k\alpha u' = 0, \quad (3.16)$$

And integrating yields, (22) and (23)

$$-k\beta u - bk^3\alpha u''' + 4bkuv = 0, \quad (3.17)$$

$$kv - k\alpha u = 0, \quad (3.18)$$

The substitutions of  $v = \alpha u$  into (24) yields

$$-k\beta u - bk^3\alpha u'' + 4bk\alpha u^2 = 0. \quad (3.19)$$

Balancing  $u^2$  with  $u''$  the gives  $n = 2$ .

According to method we assume that (26) has the solution

$$u(\xi) = A_0 + A_1 \sinh(w) + B_1 \cosh(w) + A_2 \sinh(w) \cosh(w) + B_2 \cosh^2(w), \quad (3.20)$$

Substituting (27) and (26) along with (4) and (5), yields a hyperbolic polynomial about

$$w'^s \sinh^i(w) \cosh^j(w) \quad (i = 0, 1; s = 0, 1; j = 0, 1, 2, \dots). \quad (3.21)$$

Setting the coefficients of (28) to zero, we get the following of equations:

$$\begin{aligned} & -k\beta A_0 - 2bk^3\alpha B_2 + 2bk^3\alpha B_2c + 4bk\alpha A_0^2 - 4bk\alpha A_1^2 = 0, \\ & bk^3\alpha A_1 - bk^3\alpha A_1c + 8bk\alpha A_0A_1 - k\beta A_1 = 0, \\ & -k\beta B_1 + 2bk^3\alpha B_1 - bk^3\alpha B_1c + 8bk\alpha A_0B_1 - 8bk\alpha A_1A_2 = 0, \\ & -k\beta A_1 + 5bk^3\alpha A_2 - 4bk^3\alpha A_2c + 8bk\alpha A_1B_1 - 8bk\alpha A_0A_2 = 0, \\ & -K\beta B_2 + 8bk^3\alpha B_2 - 4bk^3\alpha B_2c + 4bk\alpha A_1^2 - 4bk\alpha A_2^2 \\ & + 8bk\alpha A_0B_2 + 4bk\alpha B_1^2 = 0, \\ & -2bk^3\alpha A_1 + 8bk\alpha A_1B_2 + 8bk\alpha B_1A_2 = 0, \\ & -2bk^3\alpha B_1 + 8bk\alpha A_1A_2 + 8bk\alpha B_1B_2 = 0, \\ & -6bk^3\alpha A_2 + 8bk\alpha A_2B_2 = 0, \\ & -6bk^3\alpha B_2 + 4bk\alpha A_2^2 + 4bk\alpha B_2^2 = 0. \end{aligned}$$

Solving equations with Maple, we derive the following solutions :

$$\begin{aligned} A_0 &= \frac{\beta}{8b\alpha} - \frac{5}{8}k^2 + \frac{1}{2}k^2c, \quad A_1 = 0, \quad B_1 = 0, \\ A_2 &= -\frac{3}{4}\alpha k^2, \quad B_2 = \frac{3}{4}k^2, \quad \beta = -4b\alpha k^2 \sqrt{\frac{1}{16} + c^2 - c} \end{aligned} \quad (3.22)$$



We have obtained solutions of (24) and (25) if  $v = \alpha u$ , these solutions are

$$\left\{ \begin{array}{l} u_{21} = \left( \frac{\beta}{8b\alpha} - \frac{5}{8}k^2 + \frac{1}{2}k^2 \right) - \frac{3}{4}k^2 cs(k(x + \alpha y - \beta t), m) \\ \quad ns(k(x + \alpha y - \beta t), m) + \frac{3}{4}k^2 ns^2(k(x + \alpha y - \beta t), m), \\ v_{21} = \left( \frac{\beta}{8b\alpha} - \frac{5}{8}k^2 + \frac{1}{2}k^2 \right) - \frac{3}{4}k^2 \alpha cs(k(x + \alpha y - \beta t), m) \\ \quad ns(k(x + \alpha y - \beta t), m) + \frac{3}{4}k^2 \alpha ns^2(k(x + \alpha y - \beta t), m) \end{array} \right. \quad (3.23)$$

When  $m \rightarrow 1$ ,  $cs(\xi, m) \rightarrow \csc h(\xi)$  and  $ns(\xi, m) \rightarrow \coth(\xi)$ ,  $c \rightarrow 0$ . so we obtain the following soliton solutions of (1) and (2). (figure 3)

$$\left\{ \begin{array}{l} u_{22} = -\frac{3}{4}k^2 - \frac{3}{4}k^2 csch(k(x + \alpha y + b\alpha k^2 t)) \coth(k(x + \alpha y + b\alpha k^2 t)) \\ \quad + \frac{3}{4}k^2 \coth^2(k(x + \alpha y + b\alpha k^2 t)), \\ v_{22} = -\frac{3}{4}k^2 \alpha - \frac{3}{4}k^2 \alpha csch(k(x + \alpha y + b\alpha k^2 t)) \coth(k(x + \alpha y + b\alpha k^2 t)) \\ \quad + \frac{3}{4}k^2 \alpha \coth^2(k(x + \alpha y + b\alpha k^2 t)) \end{array} \right. \quad (3.24)$$

when  $m \rightarrow 0$ ,  $cs(\xi, m) \rightarrow \coth(\xi)$  and  $ns(\xi, m) \rightarrow \csc(\xi)$ ,  $c \rightarrow 1$  so we obtain the following trigonometric function solutions of (1) and (2)

$$\left\{ \begin{array}{l} u_{23} = -\frac{1}{4}k^2 - \frac{3}{4}k^2 \coth(k(x + \alpha y + b\alpha k^2 t)) \csc(k(x + \alpha y + b\alpha k^2 t)) \\ \quad + \frac{3}{4}k^2 \csc^2(k(x + \alpha y + b\alpha k^2 t)), \\ v_{23} = -\frac{1}{4}k^2 \alpha - \frac{3}{4}k^2 \alpha \coth(k(x + \alpha y + b\alpha k^2 t)) \csc(k(x + \alpha y + b\alpha k^2 t)) \\ \quad + \frac{3}{4}k^2 \alpha \csc^2(k(x + \alpha y + b\alpha k^2 t)) \end{array} \right. \quad (3.25)$$

Some of the properties of these solutions of (1) and (2) are shown by means of figures as follows: figure1 and figure 2 and figure 3 show the properties of  $u_{12}$ ,  $v_{12}$  and  $u_{13}$ ,  $v_{13}$  and  $u_{22}$ ,  $v_{22}$ , respectively, where we select

parameters as follows:

$$k = \frac{1}{2}, \alpha = \frac{1}{2}, b = 4$$

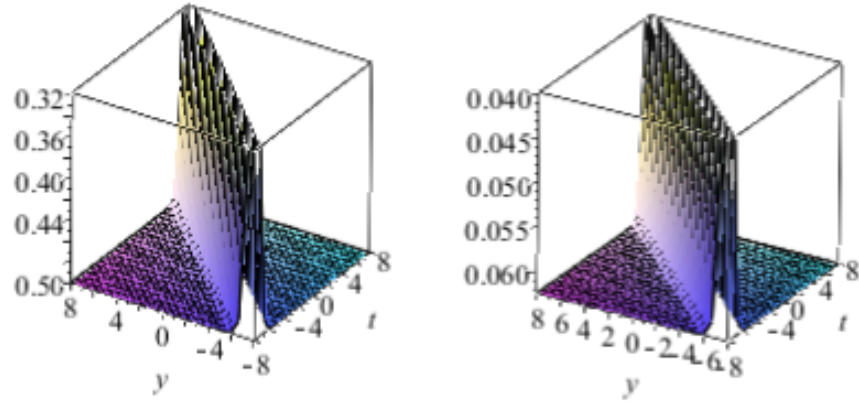


Fig. 1. the soliton solutions  $u_{12}, v_{12}$  of the (2+1)-dimensional breaking soliton equation are shown at  $x = 0$ .

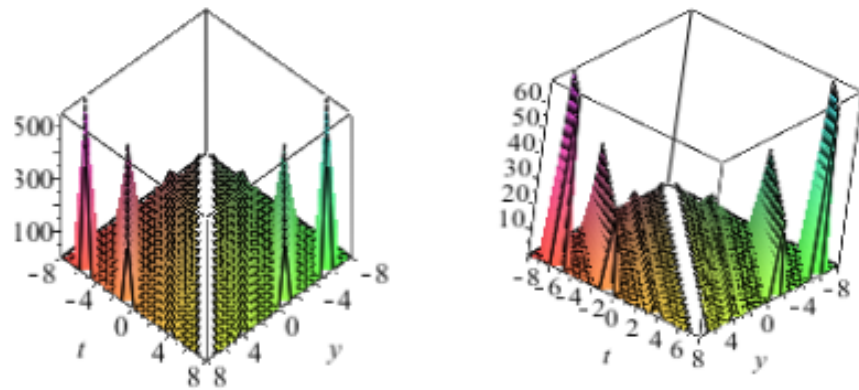


Fig. 2. trigonometric function solutions  $u_{13}, v_{13}$  of the (2+1)-dimensional breaking soliton equation are shown at  $x = 0$ .

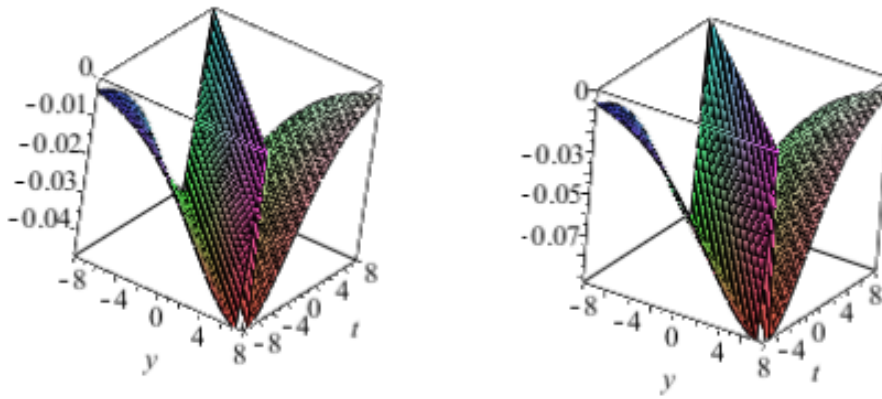


Fig. 3. the soliton solutions  $u_{22}, v_{22}$  of the (2+1)-dimensional breaking soliton equation are shown at  $x = 0$ .

In summary, we have the sinh-Gordon expansion method and sinh-cosh method to the (2+1)-dimensional breaking soliton equation. As a result, Jacobi elliptic function solutions are obtained. When  $m \rightarrow 1$ , we get the soliton solutions; while when  $m \rightarrow 0$ , we get the trigonometric function solutions.

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