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The study properties of subclass of Starlike functions

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Abstract

In this paper we introduce and investigate a certain subclass of univalent functions which are analytic in the unit disk U.Such results as coefficient inequalities. The results presented here would provide extensions of those given in earlier works.

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1 Introduction

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are analytic in the punctured open unit disk

$$U^* = \{z \in \mathbb{C} : 0 < |z| \leqslant 1\} =: U - \{0\}$$

where U is an open unit disk. Let for $0 \leqslant \alpha < 1$,

(1)

$$\Sigma^*(\alpha) = \{f \in \Sigma : Re[\frac{zf^{'}(z)}{f(z)}] < -\alpha\},$$

(2)

$$ME(\alpha) = \{f \in \Sigma : Re(zf(z)) > \alpha \mid z^2 f^{'}(z) + zf(z) \mid\},$$

(3)

$$MF(\alpha) = \{f \in \Sigma :\mid \frac{zf^{'}(z)}{f(z)} + 1 \mid< 1 - \alpha\}.$$

For $\alpha = 0$, we take $\Sigma(0) = \Sigma^*$ and ME(0) = ME and for $\alpha = 1$, we take $MF(0) = MF$.
For some recent investigations on analytic starlike functions, see (for example) the earlier works [6] and the references cited in each of these earlier investigations.

Lemma 1.1 *Let $h(z) = 1 + b_1 z^1 + b_2 z^2 + ...$ be analytic in the open unit disk U and $f(z) = \frac{h(z)}{z}$. Then*

$$1) f \in \Sigma^*(\alpha) \Leftrightarrow Re[\frac{zh^{'}(z)}{h(z)}] < 1 - \alpha,$$

$$2) f \in ME(\alpha) \Leftrightarrow Re[h(z)] < \alpha \mid zh^{'}(z) \mid,$$

$$3) f \in MF(\alpha) \Leftrightarrow \mid \frac{zh^{'}(z)}{h(z)} \mid < 1 - \alpha.$$

Definition 1.1 *Denote by* Λ *the class of functions*

$$h(z) = 1 + \Sigma_{n=1}^{\infty} b_n z^n = 1 + b_1 z^1 + b_2 z^2 + ... \qquad (1.1)$$

which are analytic in the open unit disk U*. Further suppose for* $0 \leqslant \alpha < 1$,

$$i)\Lambda^*(\alpha) = \{h \in \Lambda : Re[\frac{zh^{'}(z)}{h(z)}] < 1 - \alpha\},$$

$$ii)\Lambda E(\alpha) = \{h \in \Lambda : Re[h(z)] > \alpha \mid zh^{'}(z) \mid\},$$

$$iii)\Lambda F(\alpha) = \{h \in \Lambda : \mid \frac{zh^{'}(z)}{h(z)} \mid < 1 - \alpha\}.$$

For $\alpha = 0$*, we take* $\Lambda^*(0) = \Lambda^*$ *and* $\Lambda F(0) = \Lambda F$ *and for* $\alpha = 1$*, we take* $\Lambda E(1) = \Lambda E$.

Definition 1.2 *Let* $h, k \in \Lambda$ *where* h *is given by (1.1) and* k *is given by*

$$k(z) = 1 + \Sigma_{n=1}^{\infty} c_n z^n = 1 + c_1 z^1 + c_2 z^2 + ...$$

The Hadamard product (or convolution) $h * k$ *is defined by*

$$(h * k)(z) = 1 + \Sigma_{n=1}^{\infty} b_n c_n z^n =: (k * h)(z).$$

2 Main results

We begin by proving inclusion relation between classes which are defined in the Section 1.

Lemma 2.1 *(See [5]) If the function $h \in \Lambda$ is given by (1), and satisfy the condition*

$$Re[h(z)] > 0 \quad , \quad (z \in U)$$

then

$$| b_n | \leqslant 2 \quad , \quad (n \in \mathbb{N}).$$

Theorem 2.1 *For $\alpha \geq 1$,*

$$\Lambda E(\alpha) \subseteq \Lambda F(1 - \frac{1}{\alpha}) \subseteq \Lambda^*(1 - \frac{1}{\alpha}).$$

Also that $\alpha = 1$ all inclusions are proper.

Proof.

$$h \in \Lambda E(\alpha) \Rightarrow Re[h(z)] > \alpha \mid zh^{'}(z) \mid \Rightarrow \mid h(z) \mid > \alpha \mid zh^{'}(z) \mid$$
$$\Rightarrow \mid \frac{zh^{'}(z)}{h(z)} \mid < \frac{1}{\alpha} \Rightarrow h \in \Lambda F(1 - \frac{1}{\alpha}),$$

And

$$h \in \Lambda F(1 - \frac{1}{\alpha}) \Rightarrow \mid \frac{zh^{'}(z)}{h(z)} \mid < \frac{1}{\alpha} \Rightarrow Re[\frac{zh^{'}(z)}{h(z)}] < \frac{1}{\alpha} \Rightarrow h \in \Lambda^*(1 - \frac{1}{\alpha}).$$

But for $\alpha = 1$ it is easy to see that $e^z \in \Lambda F - \Lambda E$ and $(1-z)^2 \in \Lambda^* - \Lambda F$. $\square$
We begin by a sufficient condition for a function of the form (1.1) to be in the class $\Lambda E(\alpha)$.

Theorem 2.2 *Suppose $h \in \Lambda$ is given by (1.1). If $\Sigma_{n=1}^{\infty}(\alpha n+1) \mid b_n \mid \leq 1$ then $h \in \Lambda E(\alpha)$.*

Proof. We get

$$Re[h(z)] = Re[1 + \Sigma_{n=1}^{\infty} b_n z^n] \geq 1 - \Sigma_{n=1}^{\infty} \mid b_n \mid,$$

And

$$\alpha \mid zh^{'}(z) \mid = \alpha \mid \Sigma_{n=1}^{\infty} n b_n z^n \mid \leq \alpha \Sigma_{n=1}^{\infty} n \mid b_n \mid .$$

Therefore if

$$1 - \Sigma_{n=1}^{\infty} \mid b_n \mid \geq \alpha \Sigma_{n=1}^{\infty} n \mid b_n \mid or \Sigma_{n=1}^{\infty} (\alpha n + 1) \mid b_n \mid \leq 1.$$

Hence we get our result. $\square$

Let $\Lambda E^{+}(\alpha)$ denote the subset of $\Lambda E(\alpha)$ such that all functions $h \in \Lambda E(\alpha)$ having the following form:

$$h(z) = 1 - \Sigma_{n=1}^{\infty} b_n z^n \quad , \quad b_n \geqslant 0.$$

Corollary 2.1 *A function h of the form $h(z) = 1 - \Sigma_{n=1}^{\infty} b_n z^n \quad , \quad b_n \geqslant 0$ is in $\Lambda E^{+}(\alpha)$ if and only if $\Sigma_{n=1}^{\infty} (\alpha n + 1) b_n \leqslant 1$. The result is sharp for the function $h(z)$ given by*

$$h(z) = 1 - \frac{1}{\alpha n + 1} z^n.$$

Corollary 2.2 *The extreme points of $\Lambda E^{+}(\alpha)$ are $h_0(z) = 1, h_n(z) = 1 - \frac{1}{\alpha n + 1} z^n, n \in \mathbb{N}$. And $h \in \Lambda E^{+}(\alpha)$ if and only if h can be written in the form*

$$h(z) = \Sigma_{n=1}^{\infty} c_n h_n(z) \quad , \quad c_n \geqslant 0 \quad , \quad \Sigma_{n=1}^{\infty} c_n = 1.$$

Corollary 2.3 *If $h(z) = 1 - \Sigma_{n=1}^{\infty} b_n z^n \quad , \quad b_n \geqslant 0$ is in $\Lambda E^{+}(\alpha)$, then*

$$1 - \frac{r}{1 + \alpha} \leqslant \mid h(z) \mid \leqslant 1 + \frac{r}{1 + \alpha},$$

with equality for $h(z) = 1 - \frac{1}{1+\alpha} z \quad , \quad z = r, ir$.

Theorem 2.3 *Let $h \in \Lambda$ be given by (1.1). Then $h \in \Lambda E(\alpha)$ if and only if*

$$Re[h(z) * \frac{1+z(\alpha e^{i\theta}-1)}{(1-z)^2}] > 0, \quad (For \quad z \in U, \quad \theta \in (-\pi, \pi]).$$

Proof. We get

$$h(z) + \alpha e^{i\theta} z h^{'}(z) = h(z) * [\frac{1}{1-z} + \alpha e^{i\theta} \frac{z}{(1-z)^2}] = h(z) * \frac{1+z(\alpha e^{i\theta}-1)}{(1-z)^2},$$

And

$$h \in \Lambda E(\alpha) \Leftrightarrow Re[h(z)] > \alpha \mid zh^{'}(z) \mid > -\alpha Re[e^{i\theta} z h^{'}(z)].$$

Hence we get our result. $\square$

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