



Bifurcation Problem for Biharmonic Asymptotically Linear Elliptic Equations

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Abstract

In this paper, we investigate the existence of positive solutions for the elliptic equation $\Delta^2 u + c(x)u = \lambda f(u)$ on a bounded smooth domain Ω of \mathbb{R}^n , $n \geq 2$, with Navier boundary conditions. We show that there exists an extremal parameter $\lambda^* > 0$ such that for $\lambda < \lambda^*$, the above problem has a regular solution but for $\lambda > \lambda^*$, the problem has no solution even in the weak sense. We also show that $\lambda^* = \frac{\lambda_1}{a}$ if $\lim_{t \rightarrow \infty} f(t) - at = l \geq 0$ and for $\lambda < \lambda^*$, the solution is unique but for $l < 0$ and $\frac{\lambda_1}{a} < \lambda < \lambda^*$, the problem has two branches of solutions, where λ_1 is the first eigenvalue associated to the problem.

Key words: asymptotically linear, extremal solution, stable minimal solution, regularity.

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1 Introduction and statement of main results

Consider the problem

$$(P_\lambda) \quad \begin{cases} \Delta^2 u + cu = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, Ω is a smooth bounded domain in \mathbb{R}^n , $n \geq 2$, $\lambda > 0$ and $c = c(x)$ a positive continuous function in $\overline{\Omega}$ and the conditions imposed on f are as follows:

H_1 : is a function defined in $[0, \infty)$

$$H_2 : f \text{ is } C^1, \text{ positive, nondecreasing and convex} \quad (1.1)$$

and

$$H_3 : \lim_{t \rightarrow \infty} \frac{f(t)}{t} = a \in (0, +\infty). \quad (1.2)$$

By a solution of (P_λ) we mean a function $u \in C^4(\overline{\Omega})$ satisfying (P_λ) . In the sequel we are interested only in nonnegative solutions and for which we have considered only $\lambda > 0$. From maximum principle, if u is a nonnegative solution then $u(x) > 0$ for a.e.

Problems of the form (P_λ) occur in a variety of situations. They generate equations that arise in models of combustion [7,8], thermal explosions [7], nonlinear heat generation [11], and the gravitational equilibrium of polytropic stars [5,10]. In particular, the Helmholtz problem occur in the study of electromagnetic radiation, seismology, acoustics [2,6,15]...

For $c = 0$, Abid *et al.* have studied in [1], the following problem

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Before that, in 1996 Radulescu and Minorescu have considered the

following harmonic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

and proved in [13], that there exists $0 < \lambda^* < \infty$, a critical value of the parameter λ , such that (1.4) has a minimal, positive, classical solution u_λ for $0 < \lambda < \lambda^*$ and does not have solutions for $\lambda > \lambda^*$. The value of a was crucial in the study of (1.4) and of the behavior of u_λ when λ approaches λ^* . In the case when $a = +\infty$, it is proved in [4] that a minimal weak solution u^* exists for $\lambda = \lambda^*$. In [12], Martel proves that in this case u^* is the unique weak solution of (E_{λ^*}) . Recently, Sanchon in [14] generalizes these results for the p -Laplacian.

In this paper, we study the existence of the critical bifurcation parameter λ^* , the regularity of the eventual solutions and the existence of extremal solution, this means solution for (P_{λ^*}) . We give a new proof to show that every weak solution is a classical one. We begin by introducing the following definition.

Definition 1.1 *A weak solution of (P_λ) is a function $u \in L^1(\Omega)$, $u \geq 0$ such that $f(u) \in L^1(\Omega)$ and*

$$\int_{\Omega} u \Delta^2 \zeta + \int_{\Omega} cu \zeta = \lambda \int_{\Omega} f(u) \zeta \quad (1.5)$$

for all $\zeta \in C^4(\overline{\Omega})$ with $\Delta \zeta = \zeta = 0$ on $\partial\Omega$.

We say that u is a weak super-solution of (P_λ) if " = " is replaced by " \geq " for all $\zeta \in C^4(\overline{\Omega})$, $\zeta \geq 0$ and $\Delta \zeta = \zeta = 0$ on $\partial\Omega$.

Remark 1.1 *If u is a weak solution of (P_λ) and $u \in L^\infty(\Omega)$, we say that u is regular solution. By elliptic regularity, we know that regular solutions are smooth and solve (P_λ) in the classical sense.*

Throughout the paper, we denote $\|\cdot\|_p$ the $L^p(\Omega)$ -norm for $1 \leq p \leq \infty$ and $\|\cdot\|$ the H^2 -norm given by

$$\|u\|^2 = \int_{\Omega} |\Delta u|^2.$$

For regular solutions, we introduce a notion of stability.

Definition 1.2 *A regular solution u of (P_{λ}) is said to be stable if the first eigenvalue $\eta_1(c, \lambda, u)$ of the linearized operator $L_{c, \lambda, u} = -\Delta + c - \lambda f'(u)$ given by*

$$\eta_1(c, \lambda, u) := \inf_{\varphi \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\Delta \varphi|^2 + \int_{\Omega} c \varphi^2 - \lambda \int_{\Omega} f'(u) \varphi^2}{\|\varphi\|_2^2},$$

is positive in $H_0^1(\Omega)$. In other words,

$$\lambda \int_{\Omega} f'(u) \varphi^2 \leq \int_{\Omega} |\Delta \varphi|^2 + \int_{\Omega} c \varphi^2 \quad \text{for any } \varphi \in H^2(\Omega) \cap H_0^1(\Omega). \quad (1.6)$$

If $\eta_1(c, \lambda, u) < 0$, the solution u is said to be unstable.

We denote by λ_1 the first eigenvalue of $L = \Delta^2 + c$ in Ω with Navier boundary condition and φ_1 a positive normalized eigenfunction associated, that is, such that

$$\begin{cases} \Delta^2 \varphi_1 + c \varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega \\ \varphi_1 > 0 & \text{in } \Omega \\ \varphi_1 = \Delta \varphi_1 = 0 & \text{on } \partial \Omega \\ \|\varphi_1\|_2 = 1 \end{cases} \quad (1.7)$$

Next, we let

$\Lambda := \{\lambda > 0 \text{ such that } (P_{\lambda}) \text{ admits a solution}\}$ and $\lambda^* := \sup \Lambda \leq +\infty$.

We denote

$$r_0 := \inf_{t > 0} \frac{f(t)}{t}. \quad (1.8)$$

Our first main statement asserts the existence of the critical value λ^* .

Theorem 1.1 *Let f a positive function satisfying (1.1) and (1.2). Then there exists a critical value $\lambda^* \in (0, \infty)$ such that the following properties hold true.*

- (i) *For any $\lambda \in (0, \lambda^*)$, problem (P_λ) has a minimal solution u_λ , which is the unique stable solution of (P_λ) and the mapping $\lambda \mapsto u_\lambda$ is increasing.*
- (ii) *For any $\lambda \in (0, \frac{\lambda_1}{a})$, u_λ is the unique solution of problem (P_λ) .*
- (iii) *If problem (P_{λ^*}) has a solution u , then*

$$u = u^* = \lim_{\lambda \rightarrow \lambda^*} u_\lambda,$$

and $\eta_1(c, \lambda^, u^*) = 0$.*

- (iv) *For $\lambda > \lambda^*$, the problem (P_λ) has no weak solution.*

For the next results, let

$$l := \lim_{t \rightarrow \infty} (f(t) - at). \quad (1.9)$$

We distinguish two different situations strongly depending on the sign of l .

Theorem 1.2 *Assume that $l \geq 0$. The following results hold.*

- (i) $\lambda^* = \frac{\lambda_1}{a}$.
- (ii) *Problem (P_{λ^*}) has no solution.*
- (iii) $\lim_{\lambda \rightarrow \lambda^*} u_\lambda = \infty$ uniformly on compact subsets of Ω .

Theorem 1.3 *Assume that $l < 0$ and the function c is nonnegative. Then we have.*

- (i) The critical value λ^* belongs to $(\frac{\lambda_1}{a}, \frac{\lambda_1}{r_0})$.
- (ii) (P_{λ^*}) has a unique solution u^* .
- (iii) The problem (P_λ) has an unstable solution v_λ for any $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$ and the sequence $(v_\lambda)_\lambda$ satisfies
 - (a) $\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} v_\lambda = \infty$ uniformly on compact subsets of Ω ,
 - (b) $\lim_{\lambda \rightarrow \lambda^*} v_\lambda = u^*$ uniformly in Ω .

2 Proof of Theorem 1.1

In the proof of this Theorem we shall make use of the following auxiliary results.

Lemma 2.1 *Given $g \in L^1(\Omega)$, there exists a unique $v \in L^1(\Omega)$ which is a weak solution of*

$$\begin{cases} \Delta^2 v + c v = g & \text{in } \Omega \\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

in the sense that

$$\int_{\Omega} v \Delta^2 \zeta + \int_{\Omega} c v \zeta = \int_{\Omega} g \zeta \quad (2.2)$$

for all $\zeta \in C^4(\overline{\Omega})$ with $\Delta \zeta = \zeta = 0$ on $\partial\Omega$.

Moreover, there exists a constant c_0 independent of g such that

$$\|v\|_1 \leq c_0 \|g\|_1.$$

In addition, if $g \geq 0$ a.e in Ω , then $v \geq 0$ a.e in Ω .

Proof. For the uniqueness, let v_1 and v_2 be two solutions of (2.1). Then $v = v_1 - v_2$ satisfies

$$\int_{\Omega} v(\Delta^2 \zeta + c \zeta) = 0$$

for all $\zeta \in C^4(\overline{\Omega})$ with $\Delta\zeta = \zeta = 0$. Given $\varphi \in \mathcal{D}(\Omega)$, there exist a ζ solution of

$$\begin{cases} \Delta^2 \zeta + c\zeta = \varphi & \text{in } \Omega \\ \zeta = \Delta\zeta = 0 & \text{on } \partial\Omega, \end{cases}$$

it follows that

$$\int_{\Omega} v\varphi = 0.$$

Since φ is arbitrary, we deduce that $v = 0$.

for the existence, since $f = f^+ - f^-$, we can assume that $f \geq 0$. Let $f_n(x) = \min\{f(x), n\}$, then the family $(f_n)_n$ converge to f in $L^1(\Omega)$.

Now let v_n the solution of

$$\begin{cases} \Delta^2 v_n + c v_n = f_n & \text{in } \Omega \\ v_n = \Delta v_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

The sequence $(v_n)_n$ is monotone nondecreasing. On the other hand, we have

$$\int_{\Omega} (v_k - v_l) = \int_{\Omega} (f_k - f_l)\zeta_0,$$

where ζ_0 is defined by

$$\begin{cases} \Delta^2 \zeta_0 + c\zeta_0 = 1 & \text{in } \Omega \\ \zeta_0 = \Delta\zeta_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

So

$$\int_{\Omega} |v_k - v_l| \leq c_0 \int_{\Omega} |f_k - f_l| dx,$$

and $(v_n)_n$ is a Cauchy sequence in $L^1(\Omega)$. Passing to the limit in (2.3), after multiplication by ζ , we have that $v = \lim v_n$ is a weak solution of equation (2.1). If we take $\zeta = \zeta_0$ in (2.2), we obtain

$$\|v\|_1 = \int_{\Omega} v = \int_{\Omega} f\zeta_0 \leq c_0 \|f\|_1.$$

□

Lemma 2.2 *If (P_λ) has a weak super solution \bar{u} , then there exists a weak solution u of (P_λ) such that $0 \leq u \leq \bar{u}$ and u does not depend on \bar{u} .*

Proof. We use a standard monotone iteration argument and maximum principle for the operator $-\Delta + c$. Let $u_0 = 0$ and u_{n+1} the solution of

$$\begin{cases} \Delta^2 u_{n+1} + c u_{n+1} = \lambda f(u_n) & \text{in } \Omega, \\ \Delta u_{n+1} = u_{n+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists by Lemma 1. We prove that $0 = u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq \bar{u}$ and $(u_n)_n$ converge to $u \in L^1(\Omega)$ which is a weak solution of (P_λ) . Moreover u is independent of \bar{u} by construction. \square

The existence of the critical value λ^* is a consequence of the following auxiliary result.

Lemma 2.3 *Problem (P_λ) has no solution for any $\lambda > \lambda_1/r_0$, but has at least one solution provided λ is positive and small enough.*

Proof. To show that (P_λ) has a solution, we use the Lemma 2. To this aim, let $\zeta_0 \in C^4(\bar{\Omega})$ given by (2.4). The choice of ζ_0 implies that ζ_0 is a super solution of (P_λ) for $\lambda \leq 1/f(\|\zeta_0\|_\infty)$. By lemma 2, there exist a weak solution u of (P_λ) such that $0 \leq u \leq \zeta_0$. Because $\zeta_0 \in C^4(\bar{\Omega})$, $u \in L^\infty(\Omega)$ (u is a regular solution) and then $u \in C^4(\bar{\Omega})$. It follows that problem (P_λ) has a solution for $\lambda \leq 1/f(\|\zeta_0\|_\infty)$.

Assume now that u is a solution of (P_λ) for some $\lambda > 0$. Using φ_1 given by (1.7) as a test function, we get

$$\int_{\Omega} u \Delta^2 \varphi_1 + \int_{\Omega} c u \varphi_1 = \lambda \int_{\Omega} f(u) \varphi_1$$

This yields

$$(\lambda_1 - \lambda r_0) \int_{\Omega} \varphi_1 u \geq 0.$$

Since $\varphi_1 > 0$ and $u > 0$, we conclude that the parameter λ should belong to $(0, \lambda_1/r_0)$.

This completes our proof. \square

Lemma 2.4 *Assume that the problem (P_λ) has a solution for some $\lambda \in (0, \lambda^*)$. Then there exists a minimal solution denoted by u_λ for the problem (P_λ) . Moreover, for any $\lambda' \in (0, \lambda)$, the problem $(P_{\lambda'})$ has a solution.*

Proof. Fix $\lambda \in (0, \lambda^*)$ and let u be a solution of (P_λ) . As above, we use the Lemma 2 to obtain a solution of (P_λ) , u_λ which is independent of u used as super solution (as mentioned in the proof of Lemma 2).

Since u_λ is independent of the choice of u , then it is a minimal solution.

Now, if u is a solution of (P_λ) , then u is a super solution of the problem $(P_{\lambda'})$ for any λ' in $(0, \lambda)$ and Lemma 2 completes the proof. \square

Proof of Theorem 1 (i)

First, we claim that u_λ is stable. Indeed, arguing by contradiction, we deduce that the first eigenvalue $\eta_1 = \eta_1(c, \lambda, u_\lambda)$ is non positive. Then, there exists an eigenfunction

$$\psi \in C^4(\overline{\Omega}) \quad \text{and} \quad \Delta\psi = \psi = 0 \quad \text{on} \quad \partial\Omega,$$

such that

$$\Delta^2\psi + c\psi - \lambda f'(u_\lambda)\psi = \eta_1\psi \quad \text{in} \quad \Omega \quad \text{and} \quad \psi > 0 \quad \text{in} \quad \Omega.$$

Consider $u^\varepsilon := u_\lambda - \varepsilon\psi$. Hence

$$\Delta^2 u^\varepsilon + c u^\varepsilon - \lambda f(u^\varepsilon) = -\eta_1 \varepsilon \psi + \lambda \left[f(u_\lambda) - f(u_\lambda - \varepsilon\psi) - \varepsilon f'(u_\lambda) \psi \right] = \varepsilon \psi (-\eta_1 + o_\varepsilon(1)).$$

Since $\eta_1 \leq 0$ for $\varepsilon > 0$ small enough, we have

$$\Delta^2 u^\varepsilon + c u^\varepsilon - \lambda f(u^\varepsilon) \geq 0 \quad \text{in } \Omega.$$

Then, for $\varepsilon > 0$ small enough, we use the strong maximum principle (Hopf's Lemma) to deduce that $u^\varepsilon \geq 0$. u^ε is a super solution of (P_λ) , so by Lemma 2 we obtain a solution u such that $u \leq u^\varepsilon$ and since $u^\varepsilon < u_\lambda$, then we contradict the minimality of u_λ .

Now, we show that (P_λ) has at most one stable solution. Assume the existence of another stable solution $v \neq u_\lambda$ of problem (P_λ) . Let $\varphi := v - u_\lambda$, then by maximum principle $\varphi > 0$ and from (1.6) taking φ as a test function, we have

$$\lambda \int_\Omega f'(v) \varphi^2 \leq \int_\Omega |\Delta \varphi|^2 + \int_\Omega c \varphi^2 = \int_\Omega \varphi \Delta^2 \varphi + \int_\Omega c \varphi^2 = \lambda \int_\Omega \left[f(v) - f(u_\lambda) \right] \varphi.$$

Therefore

$$\int_\Omega \left[f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) \right] \varphi \geq 0.$$

Thanks to the convexity of f , the term in the brackets is non positive, hence

$$f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) = 0 \quad \text{in } \Omega,$$

which implies that f is affine over $[u_\lambda, v]$ in Ω . So, there exists two real numbers \bar{a} and b such that

$$f(x) = \bar{a}x + b \quad \text{in } [0, \max_\Omega v].$$

Finally, since u_λ and v are two solutions to $\Delta^2 w + cw = \bar{a}w + b$, we obtain that

$$0 = \int_\Omega (u_\lambda \Delta v - v \Delta u_\lambda) = b \int_\Omega (v - u_\lambda) = b \int_\Omega \varphi.$$

This is impossible since $b = f(0) > 0$ and φ is positive in Ω .
 Finally, by Lemma 4 and the definition of u_λ , we have that the function $\lambda \rightarrow u_\lambda$ is an increasing mapping.

Proof of Theorem 1 (ii)

In this stage, we need the following results.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded open subset of \mathbb{R}^n , $n \geq 2$. Assume that f is a function satisfying (1.1) and (1.2). If (P_λ) has a weak solution u , then u is a regular solution and hence a classical solution.*

Proof. By convexity of f , we have $a = \sup_{t \geq 0} f'(t)$ and

$$f(t) \leq at + f(0) \text{ for all } t \geq 0. \quad (2.5)$$

Let u a weak solution of (P_λ) , $f(u) \in L^1(\Omega)$.
 By elliptic regularity, $u \in L^p(\Omega)$, for all $p \geq 1$ such that

$$p < \frac{n}{n-4} \quad (p \leq \infty \text{ if } n = 2, 3 \text{ and } p < \infty \text{ if } n = 4) \quad (2.6)$$

Again by (2.5), $f(u) \in L^p$ for all p satisfying (2.6) so $u \in W^{4,r}(\Omega)$ for all $r \geq 1$ such that

$$r < \frac{n}{n-8} \quad (r \leq \infty \text{ if } n = 2, 3, 4, 5, 6, 7 \text{ and } r < \infty \text{ if } n = 8) \quad (2.7)$$

By iteration and after $k(n) = \lceil \frac{n}{4} \rceil + 1$ operations, the solution u belongs to $L^\infty(\Omega)$.

By elliptic regularity and standard bootstrap argument, $u \in C^4(\overline{\Omega})$.
 \square

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded open subset of \mathbb{R}^n , $n \geq 2$. Assume that $f(t) = f_0(t) = at + b$, where $a, b > 0$. Then*

$$(i) \quad \lambda^* = \frac{\lambda_1}{a}$$

(ii) The problem (P_λ) has no weak solution for $\lambda = \lambda^*$

Proof. Let $0 < \lambda < \frac{\lambda_1}{a}$, the problem (P_λ) , given by

$$\begin{cases} \Delta^2 u + (c - \lambda a)u = \lambda b & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.8)$$

has a unique solution in $C^4(\overline{\Omega})$.

Since $\lambda a < \lambda_1$, by Maximum principle $u > 0$. Now let $\lambda = \frac{\lambda_1}{a}$.

If the problem (2.8) has a solution u , then by multiplication (2.8) by φ_1 a positive function associated to λ_1 and introduced by (1.7) and integration by parts, it follows that $\int_\Omega \varphi_1 = 0$ which is impossible since $\varphi_1 > 0$ in Ω . So for $f_0(t) = at + b$, a and $b > 0$, we have $\lambda^* = \frac{\lambda_1}{a}$ and the equation (P_{λ^*}) has no solution. \square

For the proof of Theorem 1 (ii), let $\lambda \in (0, \frac{\lambda_1}{a})$, $b = f(0)$ and w a solution for the problem (2.8) when

$f_0(t) = at + b$. Since we have for the function f in Theorem 1, $f(w) \leq aw + f(0)$, then w is a super-solution of (P_λ) and hence by Lemma 2 and Proposition 1, the equation (P_λ) has a solution.

For the uniqueness, let u a solution of (P_λ) for a reel $\lambda \in (0, \frac{\lambda_1}{a})$. We denote $\lambda_1(L)$ the first eigenvalue of an operator L , that is $\lambda_1(\Delta^2 + c) = \lambda_1$.

Because $a = \sup_{t \geq 0} f'(t)$, we have $\Delta^2 + c - \lambda f'(u) \geq \Delta^2 + c - \lambda a$ and so

$$\lambda_1(\Delta^2 + c - \lambda f'(u)) \geq \lambda_1(\Delta^2 + c - \lambda a)$$

that is

$$\eta_1(c, \lambda, u) \geq \lambda_1 - \lambda a > 0.$$

The solution u is stable then, by Theorem 1 (i), we obtain $u = u_\lambda$. \square

Proof of Theorem 1 (iii)

Suppose that (P_λ) has a solution u . then, for every $\lambda \in (0, \lambda^*)$, we have $u_\lambda \leq u$ and so $u^* = \lim_{\lambda \rightarrow \lambda^*} u_\lambda$ is well defined in $L^1(\Omega)$ and furthermore u^* is a weak then classical solution for (P_{λ^*}) .

Since $0 \leq u^* \leq u$, u^* is a minimal solution and also satisfies (1.6) for $\lambda = \lambda^*$ so $\eta_1(c, \lambda^*, u^*) \geq 0$.

Now, consider the nonlinear operator

$$\begin{aligned} G : (0, +\infty) \times C^{4,\alpha}(\bar{\Omega}) \cap E &\longrightarrow C^{0,\alpha}(\bar{\Omega}) \\ (\lambda, u) &\longmapsto \Delta^2 u + cu - \lambda f(u), \end{aligned}$$

where $\alpha \in (0, 1)$ and E the function space defined by

$$E := \{u \in H^4(\Omega) / \Delta u = u = 0 \text{ on } \partial\Omega\} \quad (2.9)$$

Assuming that the first eigenvalue $\eta_1(c, \lambda^*, u^*)$ is positive. By the implicit function theorem applied to the operator G , it follows that problem (P_λ) has a solution for λ in a neighborhood of λ^* . But this contradicts the definition of λ^* so $\eta_1(c, \lambda^*, u^*) = 0$.

Furthermore, u^* is the unique solution for (P_{λ^*}) and we can proceed as in the proof of Theorem 1.1(ii). \square

Proof of Theorem 1 (iv)

If the problem (P_λ) has a weak solution u for $\lambda > \lambda^*$, then by Proposition 2, u is a classical solution for (P_λ) and this contradicts the definition of λ^* .

3 Proof of Theorem 1.2

In the proof of Theorem 1.2, we shall use the following auxiliary result which is a reformulation of Theorem due to Hörmander [9] and maximum principle.

Lemma 3.1 *Let Ω be an open bounded subset of \mathbb{R}^n , $n \geq 2$ with smooth boundary. Let (u_n) be a sequence of nonnegative functions defined on Ω and satisfying $\Delta^2 u_n + cu_n \geq 0$ for a positive continuous function c . Then the following alternative holds.*

- (i) $\lim_{n \rightarrow \infty} u_n = \infty$ uniformly on compact subsets of Ω ,
- or*
- (ii) (u_n) contains a subsequence which converges in $L^1_{loc}(\Omega)$ to some function u .

We first prove the following result.

Proposition 3.1 *Let f be a positive function satisfying (1.1) and (1.2). Then the following assertions are equivalent.*

- (i) $\lambda^* = \frac{\lambda_1}{a}$.
- (ii) (P_{λ^*}) has no solution.
- (iii) $\lim_{\lambda \rightarrow \lambda^*} u_\lambda = \infty$ uniformly on compact subsets of Ω .

Proof.

(i) \Rightarrow (ii). By contradiction. Assume that (P_{λ^*}) has a solution u . By (ii) of Theorem 1.1, $u = u^* = \lim_{\lambda \rightarrow \lambda^*} u_\lambda$ and $\eta_1(c, \lambda^*, u^*) = 0$. Thus there exists $\psi \in C^4(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta^2 \psi + c\psi - \lambda^* f'(u^*)\psi = 0 & \text{in } \Omega \\ \psi > 0 & \text{in } \Omega \\ \Delta \psi = \psi = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

Using φ_1 given by (1.7) as a test function, we obtain

$$\int_{\Omega} (\lambda_1 - \lambda^* f'(u^*)) \varphi_1 \psi = 0 \quad (3.2)$$

Since $\varphi_1 > 0$, $\psi > 0$, $\lambda^* = \frac{\lambda_1}{a}$, and $a = \sup_{t>0} f'(t)$, we have

$$\lambda_1 - \lambda^* f'(u^*) \geq 0.$$

Then equality (3.2) gives $f'(u^*) = a$ in Ω .

This implies that $f(t) = at + b$ in $[0, \max_{\Omega} u^*]$ for some scalar $b > 0$ and this is impossible by Proposition 2.2. Hence (P_{λ^*}) has no solution.

(ii) \Rightarrow (iii). By contradiction, suppose that (iii) doesn't hold. By Lemma 3.1 and up to subsequence,

u_{λ} converges locally in $L^1(\Omega)$ to the function u^* as $\lambda \rightarrow \lambda^*$.

Claim: u_{λ} is bounded in $L^2(\Omega)$.

Indeed, if not, we may assume that

$$u_{\lambda} = k_{\lambda} w_{\lambda}$$

with

$$\int_{\Omega} w_{\lambda}^2 dx = 1 \quad \text{and} \quad \lim_{\lambda \rightarrow \lambda^*} k_{\lambda} = \infty \quad (3.3)$$

We have

$$\frac{\lambda}{k_{\lambda}} f(u_{\lambda}) \rightarrow 0 \quad \text{in} \quad L^1_{loc}(\Omega) \quad \text{as} \quad \lambda \rightarrow \lambda^*.$$

and then

$$\Delta^2 w_{\lambda} + c w_{\lambda} \rightarrow 0 \quad \text{in} \quad L^1_{loc}(\Omega). \quad (3.4)$$

We have

$$\begin{aligned} \int_{\Omega} |\Delta w_{\lambda}|^2 &= \int_{\Omega} \Delta^2 w_{\lambda} w_{\lambda} \\ &= \int_{\Omega} \left(\frac{\lambda f(u_{\lambda})}{k_{\lambda}} - c w_{\lambda} \right) w_{\lambda}, \end{aligned}$$

then

$$\begin{aligned}
\int_{\Omega} |\Delta w_{\lambda}|^2 &\leq \int_{\Omega} \frac{\lambda f(u_{\lambda})}{k_{\lambda}} w_{\lambda} \\
&\leq \lambda^* \int_{\Omega} a w_{\lambda}^2 + \frac{f(0)}{k_{\lambda}} w_{\lambda} \\
&\leq a\lambda^* + c_0 \int_{\Omega} w_{\lambda} \\
&\leq a\lambda^* + c_0 \sqrt{|\Omega|},
\end{aligned}$$

for some $c_0 > 0$ independent of λ .

Then (w_{λ}) is bounded in $H^4(\Omega)$ and up to a subsequence, we obtain

$$\begin{aligned}
w_{\lambda} &\rightharpoonup w && \text{weakly in } && H^4(\Omega) && \text{and} \\
w_{\lambda} &\rightarrow w && \text{strongly in } && L^2(\Omega) && \text{as } \lambda \rightarrow \lambda^*. \quad (3.5)
\end{aligned}$$

Moreover, by the trace Theorem

$$\Delta w_{\lambda} = w_{\lambda} = 0 \quad \text{on} \quad \partial\Omega$$

It follows by (3.5), that $w = 0$ in Ω and this contradicts (3.3).

This complete the proof of the claim.

Thus u_{λ} is bounded in $L^2(\Omega)$ and with the same argument above, u_{λ} is bounded in $H^4(\Omega)$ and up to a subsequence, we have

$$\begin{aligned}
u_{\lambda} &\rightharpoonup u^* && \text{weakly in } && H^4(\Omega) && \text{and} \\
u_{\lambda} &\rightarrow u^* && \text{in } && L^2(\Omega) && \text{as } \lambda \rightarrow \lambda^*
\end{aligned}$$

and

$$\begin{cases} \Delta^2 u^* + cu^* = \lambda^* f(u^*) & \text{in } \Omega \\ \Delta u^* = u^* = 0 & \text{on } \partial\Omega \end{cases}$$

and this impossible by the hypothesis (ii).

It's obvious that (iii) \Rightarrow (ii) and hence (ii) \Leftrightarrow (iii).

(iii) \Rightarrow (i). If (iii) occurs, that (ii) also is true and we have $\lim_{\lambda \rightarrow \lambda^*} \|u_{\lambda}\|_2 = \infty$. Let

$$u_{\lambda} = k_{\lambda} w_{\lambda} \quad \text{with} \quad \|w_{\lambda}\|_2 = 1. \quad (3.6)$$

Up to subsequence, we obtain

$$\begin{aligned} w_\lambda &\rightharpoonup w && \text{weakly in } && H^4(\Omega) \text{ and} \\ w_\lambda &\rightarrow w && \text{in } && L^2(\Omega) \text{ as } \lambda \rightarrow \lambda^*. \end{aligned} \quad (3.7)$$

We have also

$$\frac{\lambda}{k_\lambda} f(u_\lambda) \rightarrow \lambda^* a w \quad \text{as } \lambda \rightarrow \lambda^* \quad (3.8)$$

and

$$-\Delta w_\lambda + c w_\lambda \rightarrow -\Delta w + c w \quad \text{in } L^2(\Omega)$$

and then

$$\begin{cases} -\Delta w + c w = a \lambda^* w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \quad (3.9)$$

Taking φ_1 as a test function in (3.9), we obtain

$$\lambda_1 \int_{\Omega} w \varphi_1 = \int_{\Omega} w (-\Delta \varphi_1 + c \varphi_1) = \int_{\Omega} a \lambda^* w \varphi_1$$

Since $\varphi_1 > 0$ and $w > 0$ in Ω , we have $\lambda^* = \frac{\lambda_1}{a}$ and this complete the proof of Proposition 3.1.

To finish the proof of Theorem 1.2, we need only to show that $(P_{\frac{\lambda_1}{a}})$ has no solution. Assume that u is a solution of $(P_{\frac{\lambda_1}{a}})$. Since

$$l := \lim_{t \rightarrow \infty} (f(t) - at) \geq 0 \quad \text{and} \quad a = \sup_{t \geq 0} f'(t),$$

we have $l \in (0, \infty)$ and $f(t) - at \geq 0$ and

$$\Delta^2 u + cu = \frac{\lambda_1}{a} f(u) \quad \text{in } \Omega. \quad (3.10)$$

Taking φ_1 as a test function in (3.10), we get $f(u) = a u$ in Ω , which contradicts $f(0) > 0$. This concludes the proof of Theorem 1.2. \square

4 Proof of Theorem 1.3

(i) We have shown that

$$\frac{\lambda_1}{a} \leq \lambda^* \leq \frac{\lambda_1}{r_0}$$

Suppose that $\lambda^* = \frac{\lambda_1}{a}$. By Proposition 3.1, we have

$$\lim_{\lambda \rightarrow \lambda^*} u_\lambda = \infty \text{ uniformly on compact subsets of } \Omega.$$

Let u_λ be the minimal solution of (P_λ) for $\frac{\lambda_1}{a} < \lambda < \lambda^*$. Then, multiplying (P_λ) by φ_1 and integrating by parts, we obtain

$$\int_{\Omega} \varphi_1 \left(\lambda_1 u_\lambda - \lambda f(u_\lambda) \right) = \int_{\Omega} \varphi_1 \left((\lambda_1 - a\lambda)u_\lambda - \lambda(f(u_\lambda) - au_\lambda) \right) = 0 \quad (4.1)$$

and then

$$\lambda \int_{\Omega} \varphi_1 \left(f(u_\lambda) - au_\lambda \right) \geq 0 \quad (4.2)$$

Passing to the limit in the inequality (4.2) as λ tends to λ^* , we find

$$0 \leq l\lambda^* \int_{\Omega} \varphi_1 < 0,$$

which is impossible and then $\lambda^* \neq \frac{\lambda_1}{a}$.

If $\lambda^* = \frac{\lambda_1}{r_0}$, let u be a solution of problem (P_{λ^*}) which exists by Proposition 3.1. Multiplying (P_{λ^*}) by φ_1 and integrating by parts, we obtain

$$\lambda_1 \int_{\Omega} u \varphi_1 = \frac{\lambda_1}{r_0} \int_{\Omega} f(u) \varphi_1$$

that is

$$\int_{\Omega} (f(u) - r_0 u) \varphi_1 = 0$$

then $f(u) = r_0 u$ in Ω , and this contradicts the fact that $f(0) > 0$.

(ii) Since $\lambda^* > \frac{\lambda_1}{a}$, the existence of a solution to (P_{λ^*}) is assured by Proposition 3.1 and the uniqueness is given by Theorem 1.1.

(iii) In this stage, we will use the mountain pass Theorem of Ambrosetti and Rabinowitz.

Theorem 4.1 [3] *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions.*

(1) *There exist positive constants R and ρ such that*

$$J(u) \geq J(u_0) + \rho, \text{ for all } u \in E \text{ with } \|u - u_0\| = R.$$

(2) *there exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$.*

Then the functional J possesses at least a critical point. The critical value is characterized by

$$\alpha := \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where

$$\Gamma := \left\{ g \in C([0, 1], E) \mid g(0) = u_0, g(1) = v_0 \right\}$$

and satisfies

$$\alpha \geq J(u_0) + \rho.$$

Let

$$\begin{aligned} J : E &\longrightarrow \mathbb{R} \\ u &\longmapsto \frac{1}{2} \int_{\Omega} |\Delta u|^2 + \frac{1}{2} \int_{\Omega} cu^2 - \int_{\Omega} F(u), \end{aligned}$$

where E is the function space defined by (2.9) and

$$F(t) = \lambda \int_0^t f(s) ds, \text{ for all } t \geq 0.$$

We take u_0 as the stable solution u_λ for each $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$.

The energy functional J belongs to $C^1(E, \mathbb{R})$ and

$$\langle J'(u), v \rangle = \int_{\Omega} \Delta u \cdot \Delta v + \int_{\Omega} cuv - \lambda \int_{\Omega} f(u)v,$$

for all $u, v \in E$.

Since $\eta_1(c, \lambda, u_\lambda) \geq 0$, the function u_λ is a local minimum for J . In order to transform it into a local strict minimum, consider the perturbed functional J_ε defined by

$$\begin{aligned} J_\varepsilon : E &\longrightarrow \mathbb{R} \\ u &\longmapsto J(u) + \frac{\varepsilon}{2} \int_{\Omega} |\Delta(u - u_\lambda)|^2 + \frac{\varepsilon}{2} \int_{\Omega} c|u - u_\lambda|^2, \end{aligned} \quad (4.3)$$

for all $\varepsilon \in [0, \varepsilon_0]$, where

$$\varepsilon_0 := \frac{3}{4} \frac{\lambda a - \lambda_1}{\lambda_1}.$$

We observe that J_ε is also in $C^1(E, \mathbb{R})$ and

$$\langle J'_\varepsilon(u), v \rangle = \int_{\Omega} \Delta u \Delta v + \int_{\Omega} cuv - \lambda \int_{\Omega} f(u)v + \varepsilon \int_{\Omega} \Delta(u - u_\lambda) \Delta v + \varepsilon \int_{\Omega} c(u - u_\lambda)v$$

for all $u, v \in E$. Using the same arguments of Mironescu and Rădulescu in [13, Lemma 9], we show that J_ε satisfies the Palais-Smale condition and so we have the next lemma.

Lemma 4.1 *Let $(u_n) \subset E$ be a Palais-Smale sequence, that is,*

$$\sup_{n \in \mathbb{N}} |J_\varepsilon(u_n)| < +\infty, \quad (4.4)$$

$$\|J'_\varepsilon(u_n)\|_{E^*} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.5)$$

Then (u_n) is relatively compact in E .

Now, we need only to check that the two geometric assumptions are fulfilled. First, since u_λ is a local minimum of J , there exists

$R > 0$ such that for all $u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $J(u) \geq J(u_\lambda)$. Then

$$J_\varepsilon(u) \geq J_\varepsilon(u_\lambda) + \frac{\varepsilon}{2} \int_\Omega |\Delta(u - u_\lambda)|^2.$$

Since $u - u_\lambda$ is not harmonic, we can choose

$$\rho := \frac{\varepsilon R^2}{4} > 0$$

and u_λ becomes a strict local minimal for J_ε , which proves (1). Also, we have

$$J_\varepsilon(t\varphi_1) = \frac{\lambda_1}{2}t^2 + \frac{\varepsilon}{2}\lambda_1 t^2 - \varepsilon\lambda_1 t \int_\Omega \varphi_1 u_\lambda + \frac{\varepsilon}{2}\lambda \int_\Omega f(u_\lambda)u_\lambda - \int_\Omega F(t\varphi_1), \quad \forall t > 0. \quad (4.6)$$

Recall that $\lim_{t \rightarrow +\infty} (f(t) - at)$ is finite, then there exists $\beta \in \mathbb{R}$ such that

$$f(t) \geq at + \beta, \quad \forall t > 0.$$

Hence

$$F(t) \geq \frac{a\lambda}{2}t^2 + \beta\lambda t, \quad \forall t > 0.$$

This yields

$$\frac{J_\varepsilon(t\varphi_1)}{t^2} \leq \left(\frac{\lambda_1}{2} + \frac{\varepsilon\lambda_1}{2} - \frac{a\lambda}{2} \right) + \frac{\varepsilon}{2t^2} \int_\Omega f(u_\lambda)u_\lambda,$$

which implies that

$$\limsup_{t \rightarrow +\infty} \frac{1}{t^2} J_\varepsilon(t\varphi_1) \leq \left(\frac{\lambda_1}{2} + \frac{\varepsilon_0\lambda_1}{2} - \frac{a\lambda}{2} \right) < 0, \quad \forall \varepsilon \in [0, \varepsilon_0].$$

Therefore

$$\lim_{t \rightarrow +\infty} J_\varepsilon(t\varphi_1) = -\infty$$

and so, $\forall \varepsilon \in [0, \varepsilon_0]$, there exists $v_0 \in E$ such that

$$J_\varepsilon(v_0) \leq J_\varepsilon(u_\lambda)$$

and (2) is proved. Finally, let v_ε (respectively. c_ε) be the critical point (respectively. critical value) of J_ε .

Remark 4.1 *The fact that J_ε increases with ε implies that for all $\varepsilon \in [0, \varepsilon_0]$, $c_\varepsilon \in [c_0, c_{\varepsilon_0}]$. Then, c_ε is uniformly bounded. Thus, for all $\varepsilon \in [0, \varepsilon_0]$, the critical point v_ε satisfies $\|v_\varepsilon - u_\lambda\| \geq R$.*

Recall that for any $\varepsilon \in [0, \varepsilon_0]$, the function v_ε belongs to E and satisfies

$$\Delta^2 v_\varepsilon + c v_\varepsilon = \frac{\lambda}{1 + \varepsilon} f(v_\varepsilon) + \frac{\lambda \varepsilon}{1 + \varepsilon} f(u_\lambda) \text{ in } \Omega, \quad (4.7)$$

and

$$J_\varepsilon(v_\varepsilon) = c_\varepsilon. \quad (4.8)$$

By Lemma 3.1, Remark 2, (4.7) and (4.8), there exists $v \in E$ such that

$$v_\varepsilon \rightarrow v \text{ in } E, \text{ as } \varepsilon \rightarrow 0,$$

satisfying

$$\Delta^2 v + c v = \lambda f(v) \text{ in } \Omega.$$

From Remark 4.1, we see that $v \neq u_\lambda$.

Proof of (a). By contradiction, suppose that (a) doesn't hold. By Lemma 3.1 there is a sequence of positives scalars (μ_n) and a sequence (v_n) of unstable solutions to (P_{μ_n}) such that $v_n \rightarrow v$ in $L^1_{loc}(\Omega)$ as $\mu_n \rightarrow \lambda_1/a$ for some function v .

We first claim that (v_n) cannot be bounded in E . Otherwise, let $w \in E$ be such that, up to a subsequence,

$$v_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad v_n \rightarrow w \text{ strongly in } L^2(\Omega).$$

Therefore,

$$\Delta^2 v_n + c v_n \rightarrow \Delta^2 w + c w \text{ in } \mathcal{D}'(\Omega) \quad \text{and} \quad f(v_n) \rightarrow f(w) \text{ in } L^2(\Omega),$$

which implies that $\Delta^2 w + c w = \frac{\lambda_1}{a} f(w)$ in Ω . It follows that $w \in E$ and solves $(P_{\lambda_1/a})$. From Lemma 4.1, we deduce that

$$\eta_1 \left(c, \frac{\lambda_1}{a}, w \right) \leq 0. \quad (4.9)$$

Relation (4.9) shows that $w \neq u_{\lambda_1/a}$ which contradicts the fact that $(P_{\lambda_1/a})$ has a unique solution. Now, since $\Delta^2 v_n + cv_n = \mu_n f(v_n)$, the unboundedness of (v_n) in E implies that this sequence is unbounded in $L^2(\Omega)$, too. To see this, let

$$v_n = k_n w_n, \quad \text{where } k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \rightarrow \infty.$$

Then

$$\Delta^2 w_n + cw_n = \frac{\mu_n}{k_n} f(v_n) \rightarrow 0 \quad \text{in } L^1_{loc}(\Omega).$$

So, we have convergence also in the sense of distributions and (w_n) is seen to be bounded in E with standard arguments. We obtain

$$\Delta^2 w + cw = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

The desired contradiction is obtained since $w \in E$.

Proof of (b). As before, it is enough to prove the $L^2(\Omega)$ boundness of v_λ near λ^* and to use the uniqueness property of u^* . Assume that $\|v_n\|_2 \rightarrow \infty$ as $\mu_n \rightarrow \lambda^*$, where v_n is a solution to (P_{μ_n}) . We write again $v_n = l_n w_n$. Then,

$$\Delta^2 w_n + cw_n = \frac{\mu_n}{l_n} f(v_n). \quad (4.10)$$

The fact that the right-hand side of (4.10) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in E . Let (w_n) be such that (up to a subsequence)

$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \rightarrow w \text{ strongly in } L^2(\Omega).$$

A computation already done shows that

$$\Delta^2 w + cw = \lambda^* aw, \quad w \geq 0 \quad \text{and} \quad \|w\|_2 = 1,$$

which forces λ^* to be λ_1/a . This contradiction concludes the proof. \square

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