Theory of Approximation and Applications

Vol. 11, No. 1, (2016), 1-12

Homotopy Perturbation Method and Aboodh Transform for Solving Nonlinear Partial Differential Equations

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Received 12 March 2015; accepted 19 April 2016

Abstract

Here, a new method called Aboodh transform homotopy perturbation method (ATHPM) is used to solve nonlinear partial differential equations, we present a reliable combination of homotopy perturbation method and Aboodh transform to investigate some nonlinear partial differential equations. The nonlinear terms can be handled by the use of homotopy perturbation method. The results show the efficiency of this method. Aboodh transform was introduced by Khalid Aboodh to facilitate the process of solving ordinary and partial differential equations in the time domain.

Key words: Aboodh transform, homotopy perturbation method, nonlinear partial differential equations.

2010 AMS Mathematics Subject Classification : 65H20, 35R11.

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1 Introduction

Linear and nonlinear partial differential equations are of fundamental importance in science and engineering. Some integral transform method such as Laplace and Fourier and Sumudu transforms methods see [1,10], are used to solve linear partial differential equations and use fullness of these integral transform lies in their ability to transform differential equations into algebraic equations which allows simple and systematic solution procedures. However, using integral transform in nonlinear differential equations may increase its complexity. In recent years, many research workers have paid attention to find the solutions of nonlinear differential equations by using various methods. Among these are the Adomian decomposition method the tanh method, the homotopy perturbation method [11–17], the differential transform method and the variational iteration method. Elzaki transform [18–22] Homotopy Perturbation and Elzaki Transform for Solving Nonlinear Partial Differential Equations [23] is totally incapable of handling the nonlinear equations because of the difficulties that are caused by the nonlinear terms. Also see [2–9]. Various ways have been proposed recently to deal with these nonlinearities, one of these combinations of homotopy perturbation method and Aboodh transform which is studies in this paper. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear partial differential equations. This article considers the effectiveness of the homotopy perturbation Aboodh transform method in solving nonlinear partial differential equations both homogeneous and non-homogeneous.

1.1 Aboodh Transform

The basic definitions of modified of Aboodh transform is defined as follows, Aboodh transform of the function $f(t)$ is

$$A\{f(t)\} = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt. \tag{1.1}$$

In this paper, we combined Aboodh transform and homotopy perturbation to solve nonlinear partial differential equations. To obtain Aboodh transform of partial derivative we use integration by parts, and then we have:

$$A\{\frac{\partial f}{\partial t}(x,t)\} = vK(x,v) - \frac{f(x,0)}{v},$$

$$A\{\frac{\partial^2 f}{\partial t^2}(x,t)\} = v^2K(x,v) - \frac{1}{v}\frac{\partial f}{\partial t}(x,0) - f(x,0),$$

$$A\{\frac{\partial f}{\partial x}(x,t)\} = \frac{d}{dx}\{K(x,v)\},$$

$$A\{\frac{\partial^2 f}{\partial x^2}(x,t)\} = \frac{d^2}{dx^2}\{K(x,v)\}.$$

Proof:
To obtain Aboodh transform of partial derivatives we use integration by parts as follows:

$$A\{\frac{\partial f}{\partial t}(x,t)\} = \int_0^\infty \frac{1}{v}\frac{\partial f}{\partial t}(x,t)e^{-vt}dt = \lim_{p\to\infty}\int_0^p \frac{1}{v}\frac{\partial f}{\partial t}(x,t)e^{-vt}dt$$

$$= \lim_{p\to\infty}\left(\left[\frac{1}{v}f(x,t)e^{-vt}\right]_0^p + \int_0^p f(x,t)e^{-vt}dt\right) = vK(x,v) - \frac{f(x,0)}{v}.$$

We assume that f is piecewise continuous and it is of exponential order.

$$A\{\frac{\partial f}{\partial x}(x,t)\} = \int_0^\infty \frac{1}{v}\frac{\partial f}{\partial x}(x,t)e^{-vt}dt = \frac{\partial}{\partial x}\int_0^\infty \frac{1}{v}f(x,t)e^{-vt}dt = \frac{d}{dx}\{K(x,v)\},$$

(using the Leibnitz rule)

By the same method we find: $A\{\frac{\partial^2 f}{\partial x^2}(x,t)\} = \frac{d^2}{dx^2}\{K(x,v)\}$.

To find $A\{\frac{\partial^2 f}{\partial t^2}(x,t)\}$, let $\frac{\partial f}{\partial t} = g$, then we have

$$A\{\frac{\partial^2 f}{\partial t^2}(x,t)\} = A\{\frac{\partial g}{\partial t}(x,t)\} = vA\{g(x,t)\} - \frac{g(x,0)}{v},$$

$$A\{\frac{\partial^2 f}{\partial t^2}(x,t)\} = v^2K(x,v) - \frac{1}{v}\frac{\partial f}{\partial t}(x,0) - f(x,0).$$

We can easily extend this result to the n^{th} partial derivative by using mathematical induction.

Let X and Y be the topological spaces. If f and g are continuous maps of the space X into Y, it is said that f is homotopic to g , if there is continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for each $x \in X$, then the map is called homotopy between f and g. To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0 \tag{1.2}$$

Where L is any differential operator, we define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u) \tag{1.3}$$

Where $F(u)$is a functional operator with known solution v_0 which can be obtained easily. It is clear that, for

$$H(u,p) = 0, \tag{1.4}$$

where $H(u,0) = F(u), \quad H(u,1) = L(u)$.
In topology this show that $H(u,p)$ continuously traces an implicitly defined carves from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The HPM uses the embed ling parameter p as a small parameter and write the solution as a power series

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + ..., \tag{1.5}$$

If $p \to 1$, then (5) corresponds to (3)and becomes the approximate solution of the form,

$$f = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i, \tag{1.6}$$

We assume that (6) has a unique solution. The comparisons of like powers of (6) give solutions of various orders.

2 Aboodh Transform Homotopy Perturbation Method

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$\begin{aligned} Du(x,t) + Ru(x,t) + Nu(x,t) &= g(x,t) \\ u(x,0) = h(x), \quad u_t(x,0) &= f(x), \end{aligned} \tag{2.1}$$

Where D is linear differential operator of order two, R is linear differential operator of less order than , N is the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking Aboodh transform on both sides of equation (7), to get:

$$A\{Du(x,t)\} + A\{Ru(x,t)\} + A\{Nu(x,t)\} = A\{g(x,t)\}. \tag{2.2}$$

Using the differentiation property of Aboodh transforms and above initial conditions, we have:

$$A\{u(x,t)\} = \frac{1}{v^2}A\{g(x,t)\}+\frac{1}{v^2}A\{h(x)\}+\frac{1}{v^3}A\{f(x)\}-\frac{1}{v^2}A\{Ru(x,t)+Nu(x,t)\}. \tag{2.3}$$

Applying the inverse Aboodh transform on both sides of equation (9), to find:

$$u(x,t) = G(x,t) - A^{-1}\left(\frac{1}{v^2}A\{Ru(x,t) + Nu(x,t)\}\right). \tag{2.4}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. So that

$$G(x,t) = A^{-1}\{\frac{1}{v^2}A\{g(x,t)\}\} + h(x) + tf(x).$$

Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{2.5}$$

And the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{2.6}$$

where $H_n(u)$ are given by:

$$H_m(u_0 m u_1, u_2, ...U_n) = \frac{1}{n!}\frac{\partial p^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0, 1, 2, ... \quad (2.7)$$

Substituting equations (11) and (12) in equation (10), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(A^{-1}\left(\frac{1}{v^2}A\{R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\}\right)\right). \quad (2.8)$$

This is the coupling of the Aboodh transform and the homotopy perturbation method. Comparing the coefficient of like powers of p, the following approximations are obtained.

$$\begin{aligned}
p^0 &: u_0(x,t) = G(x,t), \\
p^1 &: u_1(x,t) = A^{-1}\left(\frac{1}{v^2}A\{Ru_0(x,t) + H_0(u)\}\right), \\
p^2 &: u_2(x,t) = A^{-1}\left(\frac{1}{v^2}A\{Ru_1(x,t) + H_1(u)\}\right), \\
p^3 &: u_3(x,t) = A^{-1}\left(\frac{1}{v^2}A\{Ru_2(x,t) + H_2(u)\}\right), \\
&\vdots
\end{aligned}$$

Then the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ... \quad (2.9)$$

3 Applications

In this section we apply the homotopy perturbation Aboodh transform method for solving nonlinear partial differential equations.

3.1 Example 1

Consider the following homogeneous nonlinear partial differential equations

$$u_t + uu_x = 0, \quad u(x,0) = -x. \tag{3.1}$$

Taking Aboodh transform of equation (16) subject to the initial condition, we have:

$$A\{u(x,t)\} = -\frac{1}{v^2}x - \frac{1}{v}A\{uu_x\}. \tag{3.2}$$

The inverse Aboodh transform implies that:

$$u(x,t) = -x - A^{-1}\{\frac{1}{v}A\{uu_x\}\}. \tag{3.3}$$

Now applying the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = -x - p\left(A^{-1}\{\frac{1}{v}A\{\sum_{n=0}^{\infty} p^n H_n(x,t)\}\}\right). \tag{3.4}$$

Where $H_n(u)$ are He's polynomials that represents the nonlinear terms. Or

$$\begin{aligned} p[uu_x] = 0 &\Rightarrow p(u_0 + pu_1 + p^2u_2 + ...)(u_0 + pu_{1_x} + p^2u_{2_x} + ...), \\ u &= u_0 + pu_{1_x} + p^2u_{2_x} + ..., \end{aligned}$$

The first few components of He's polynomials, are given by

$$\begin{aligned} H_0 &= u_0u_{0_x}, \\ H_1 &= u_{0_x}u_1 + u_0u_{1_x}, \\ H_2 &= u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0, \end{aligned}$$

Comparing the coefficients of the same powers of p, we get:

$$\begin{aligned} p^0 &: u_0(x,t) = -x, \\ p^1 &: u_1(x,t) = -A^{-1}\{\frac{1}{v}A\{H_0(x,t)\}\} = -xt, \quad H_1(u) = 2xt \\ p^2 &: u_2(x,t) = -A^{-1}\{\frac{1}{v}A\{H_1(x,t)\}\} = -xt^2, \quad H_2(u) = 3x^2t \\ p^3 &: u_3(x,t) = -A^{-1}\{\frac{1}{v}A\{H_2(x,t)\}\} = -xt^3, \end{aligned}$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = -x - xt - xt^2 - xt^3 - ... = -x(1+t+t^2+t^3+...) = \frac{x}{t-1}.$$

3.2 Example 2

Consider the first order nonlinear partial differential equation

$$u_t + uu_x = 2t + x + t^3 + xt^2, \qquad u(x,0) = -x. \tag{3.5}$$

To find the solution by homotopy perturbation Aboodh transform method, we applying homotopy perturbation method after taking Aboodh and inverse Aboodh transforms of equation (20), we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - p\left(-A^{-1}\{\frac{1}{v^2}A\{\sum_{n=0}^{\infty} p^n H_n(x,t)\}\}\right), \tag{3.6}$$

where,

$$\begin{aligned} H_0 &= u_0 u_{0_x}, \\ H_1 &= u_{0_x} u_1 + u_0 u_{1_x}, \\ H_2 &= u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \end{aligned}$$

Comparing the coefficients of like powers of p, we have:

$$\begin{aligned} p^0 &: u_0(x,t) = t^2 + xt + \frac{xt^3}{3} + \frac{t^4}{4}, \\ p^1 &: u_1(x,t) = \frac{xt^3}{3} - \frac{t^4}{4} - 2\frac{xt^5}{15} - \frac{7t^6}{72} - \frac{xt^7}{63} - \frac{t^8}{98}, \\ &\vdots \end{aligned}$$

The noise terms appear between the components $u_0(x,t)$ and $u_1(x,t)$, therefore, the exact solution is given by

$$u(x,t) = t^2 + xt.$$

3.3 Example 3

Let us consider the second order nonlinear partial differential equation

$$u_t = u_x^2 + uu_{xx}, \qquad u(x,0) = x^2 \tag{3.7}$$

As applied in the previous examples, we thus obtain

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 - p\left(A^{-1}\{\frac{1}{v^2}A\{\sum_{n=0}^{\infty} p^n H_n(u)\}\}\right),$$

where $H(u) = u_x^2 + uu_{xx}$, with few terms as

$$\begin{aligned} H_0 &= u_{0_x}^2 + u_0 u_{0_{xx}}, \\ H_1 &= 2u_{0_x}u_1 + u_0 u_{1_{xx}} + u_1 u_{0_{xx}}, \\ &\vdots \end{aligned}$$

$$\begin{aligned} p^0 &: u_0(x,t) = x^2, \quad H_0(u) = 6x^2 \\ p^1 &: u_1(x,t) = A^{-1}\{\frac{1}{v}A\{H_0(u)\}\} = 6x^2t, \quad H_1(u) = 72x^2t \\ p^2 &: u_2(x,t) = A^{-1}\{\frac{1}{v}A\{H_1(u)\}\} = 36x^2t^2, \\ &\vdots \end{aligned}$$

Then the solution of equation (22) is given by:

$$u(x,t) = x^2(1 + 6t + 36t^2 + ...) = \frac{x^2}{1-6t}. \tag{3.8}$$

4 Conclusion

In this paper, we mix the Aboodh transform and homotopy perturbation method to solve nonlinear partial differential equations. The solution by using Adomian decomposition method is simple, but the calculation of Adomian polynomials is complex. The fact that the developed algorithm solves nonlinear partial differential equations without Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

Acknowledgment

The author would like to thank the anonymous reviewer for his/her valuable comments.

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