On co-Farthest Points in Normed Linear Spaces

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Abstract

In this paper, we consider the concepts co-farthest points in normed linear spaces. At first, we define farthest points, farthest orthogonality in normed linear spaces. Then we define co-farthest points, co-remotal sets, co-uniquely sets and co-farthest maps. We shall prove some theorems about co-farthest points, co-remotal sets. We obtain a necessary and coefficient conditions about co-farthest points and dual spaces.

Key words: Farthest points, Co-farthest points, Co-farthest map.

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1 Introduction

A kind of approximation, called best co-approximation was introduced by Franchettei and Furi in 1972 [12]. Some results on best co-approximation theory in linear normed spaces have been obtained by P. L. Papini and I. Singer [35]. In this section we consider co-proximinality and co-remotality in normed linear spaces.

**Definition 1.1** Let \((X, \|\cdot\|)\) be a normed linear space, \(G\) a non-empty subset of \(X\) and \(x \in X\). We say that \(g_0 \in G\) is a best co-approximation of \(x\) whenever \(\|g - g_0\| \leq \|x - g\|\) for all \(g \in G\). We denote the set of all best co-approximations of \(x\) in \(G\) by \(R_G(x)\).

We say that \(G\) is a co-proximinal subset of \(X\) if \(R_G(x)\) is a non-empty subset of \(G\) for all \(x \in X\). Also, we say that \(G\) is a co-Chebyshev subset of \(X\) if \(R_G(x)\) is a singleton subset of \(G\) for all \(x \in X\).

**Definition 1.2** Let \((X, \|\cdot\|)\) be a normed linear space, \(A\) a subset of \(X\), \(x \in X\) and \(m_0 \in A\). We say that \(m_0\) is co-farthest to \(x\) if \(\|m_0 - a\| \geq \|x - a\|\) for every \(a \in A\). The set of co-farthest points to \(x\) in \(A\) is denoted by

\[ C_A(x) = \{a_0 \in A : \|a_0 - a\| \geq \|x - a\| \text{ for every } a \in A \backslash \{a_0\}\}. \]

The set \(A\) is said to be co-remotal if \(C_A(x)\) has at least one element for every \(x \in X\). If for each \(x \in X\), \(C_A(x)\) has exactly one element in \(A\), then the set \(A\) is called co-uniquly remotal. We define for \(a_0 \in A\),

\[ C_A^{-1}(a_0) = \{x \in X : \|a_0 - a\| \geq \|x - a\| \text{ for every } a \in A\}. \]

\(C_A^{-1}(a_0)\) is a closed set and \(a_0 \in C_A^{-1}(a_0)\). Note that if \(x \in A\), then \(x \in C_A(x)\).

**Example 1.1** Suppose \(X = \mathbb{R}\) and \(A = [1, 2] \cup \{3\} \backslash \{1\}\). We set \(x = 1\) and \(a_0 = 3\). Then \(a_0 \in C_A(x)\).
2 Co-Probiminality, co-Chebyshevity and co-Remotality

In this section we consider co-proximinality and co-Chebyshevity and co-remotality in normed linear spaces.

**Theorem 2.1** Let \((X, \| \cdot \|)\) be a normed linear space and \(A\) a subset of \(X\).

a) If for every \(x \in X\) and for every \(a \in A\), \(a \in H_{d_x}\), then \(A\) is co-proximinal.

b) If for every \(x \in X\) and for every \(a \in A\), there exists a unique \(b \in H_{\|x-a\|}\), then \(A\) is co-Chebyshev.

**Proof.** a) Suppose \(x \in X\), for every \(a \in A\) there exists \(a_0 \in A\) such that \(a - a_0 \in B[0, d_x]\). Therefore for every \(a \in A\)

\[
\|a - a_0\| \leq d_x
\]

\[
\leq \|x - a\|
\]

That is \(a_0 \in R_A(x)\) so \(A\) is co-proximinal.

b) Suppose \(x \in X\), \(a \in A\) and there exists an unique \(b \in H_{\|x-a\|}\), by part (a), \(R_A(x)\) is non-empty. The set \(A\) is co-proximinal.

For each \(x \in X\) if there exist \(a_1, a_2 \in R_A(x)\), then for \(a \in A\) we have \(\|a_i - a\| \leq \|x - a\|\) for \(i = 1, 2\). Therefore for \(a \in A, a_i - a \in B[0, \|x - a\|]\), and for \(a \in A\), we have \(a_i \in H_{\|x-a\|}\). This is a contraction. It follows that \(A\) is co-Chebyshev.

**Theorem 2.2** Let \((X, \| \cdot \|)\) be a normed linear space and \(A\) a subset of \(X\).

a) If for every \(x \in X\) and for every \(a \in A\), \(a \in K_{\delta_x}\), then \(A\) is co-remotal.
b) If for every \( x \in X \) and for every \( a \in A \), there exists a unique \( b \in K_{\|x-a\|} \), then \( A \) is co-uniquely remotal.

**Proof.** a) Suppose \( x \in X \) and \( a \in A \). Suppose there exists an \( a_0 \in A \) such that \( a - a_0 \in B^c[0, \delta_x] \). Therefore for every \( a \in A \)

\[
\|a - a_0\| \geq \delta_x \\
\geq \|x - a\|.
\]

That is \( a_0 \in C_A(x) \) so \( A \) is co-remotal.

b) If \( x \in X \) and \( a \in A \) if there exists an unique \( b \in K_{\|x-a\|} \), then \( C_A(x) \) is non-empty. The set \( A \) is co-remotal.

For \( x \in X \) if there exist \( a_1, a_2 \in C_A(x) \), then for \( a \in A \) we have \( \|a_i - a\| \leq \|x - a\| \) for \( i = 1, 2 \). Therefore for \( a \in A \), \( a_i - a \in B^c[0, \|x - a\|] \), and for \( a \in A \), we have \( a_i \in K_{\|x-a\|} \). This is a contraction. It follows that \( A \) is co-uniquely remotal. Let \( W \) be a non-empty bounded subset of a normed linear space \((X, \|\|)\). If there exists a point \( \omega_0 \in W \) such that \( \delta(x,W) = \sup\{\|x - \omega\| : \omega \in W\} = \|x - \omega_0\| \) for \( x \in X \). Then \( \omega_0 \) is called farthest point in \( W \) from \( x \). The set of all such \( \omega_0 \in W \) is denoted by \( F_W(x) \).

**Theorem 2.3** Let \( A \) be a bounded subset of a normed linear space, \( A + A = A \), \( -A = A \) and \( 0 \in A \),

(i) If \( a_0 \in A \), then \( C_A^{-1}(a_0) = -a_0 + C_A^{-1}(0) \),

(ii) \( C_A(x) = (-x + C_A^{-1}(0)) \cap A \).

(iii) If \( a_0 \in A \), then \( x \in C_A(a_0) \) if and only if \( x - a_0 \in C_A^{-1}(a_0) \)

**Proof.** (i)
\[ x \in C_A^{-1}(a_0) \iff a_0 \in C_A(x) \]
\[ \iff \|a_0 - a\| \geq \|x - a\| \text{ for every } a \in A \setminus \{a_0\} \]
\[ \iff \|u\| \geq \|x - a_0 - u\| \text{ for every } u \in A \text{ since } A + A = A \]
\[ \iff x + a_0 \in C_A^{-1}(0) \]
\[ \iff x \in -a_0 + C_A^{-1}(0). \]

(ii)

\[ a_0 \in C_A(x) \iff x \in C_A^{-1}(a_0) \]
\[ \iff x + a_0 \in C_A^{-1}(0) \]
\[ \iff a_0 \in -x - C_A^{-1}(0) \text{ and } a_0 \in A. \]

(iii) Suppose \( x - a_0 \in C_A^{-1}(a_0) \), then
\[ \|a\| \geq \|x - a_0 - a\|. \]
Since \( A + A = A \) and \( -A = A \), then \( a - a_0 \in A + A \). Then
\[ \|b\| \geq \|x - a_0 - b\| \text{ for every } b \in A, \]

Therefore \( x - a_0 \in C_A^{-1}(a_0) \).

**Theorem 2.4** Let \( A \) be a bounded subset of a normed linear space, then the following statements are equivalent:

(i) \( A \) is co-remotal,

(ii) \( X = -A + C_A^{-1}(0) \).

**Proof.** (i) \( \rightarrow \) (ii). Suppose \( A \) is co-remotal and \( x \in X \), there exists a \( a_0 \in A \) such that \( a_0 \in C_A(x) \). Then \( u_0 = x + a_0 \in C_A^{-1}(0) \), and \( x = -a_0 + u_0 \in -A + C_A^{-1}(0) \).

(ii) \( \rightarrow \) (i). If \( X = -A + C_A^{-1}(0) \) and \( x \in X \). Then there exist a \( a_0 \in A \) such that \( x + a_0 \in C_A^{-1}(0) \). Thus \( a_0 \in C_A(x) \) and \( A \) is co-remotal.

**Theorem 2.5** Let \( A \) be a co-remotal subset of a normed linear space,
\( A = A + A \) and \( 0 \in A \), then there exists an element \( z \in X \setminus \{0\} \) such that \( 0 \in C_A(z) \).

**Proof.** Suppose \( x \in X \setminus A \), since \( A \) is co-remotal, there exists \( a_0 \in C_A(x) \) and so \( z = x + a_0 \in C_A^{-1}(0) \). Hence \( 0 \in C_A(z) \), \( z \neq 0 \).

**Theorem 2.6** Let \((X, \|\|)\) be a normed linear space, \( A \) a bounded subset of \( X \), \( x \in X \), \( A = A + A \) and \( 0 \in A \). If \( 0 \in C_A(x) \), then \( A \perp_F x \).

**Proof.** If \( 0 \in C_A(x) \) and \( a \in A \). Then \( \|a\| \geq \|x - a\| \), therefore \( A \perp_F x \).

**Theorem 2.7** Let \((X, \|\|)\) be a normed linear space and \( x, y \in X \). Then the following statements are equivalent:

(i) \( A \perp_F x \) or \( 0 \in C_A(x) \),

(ii) For every \( m \in A \), there exists an \( f \in X^* \) such that \( f \) satisfies \( \|f\| = 1 \) and \( |f(m)| \geq \delta(x, A) \).

**Proof.** (i) → (ii). Suppose \( A \perp_F x \) then for \( m \in A \), \( m \perp_F x \). That is \( \|m\| \geq \delta(x, A) \). By Hahn-Banach Theorem, there exists an \( f \in X^* \) such that \( \|f\| = 1 \) and \( |f(m)| = \|m\| \geq \delta(x, A) \).

(ii) → (i). Suppose there exists an \( f \in X^* \) such that \( f \) satisfies \( \|f\| = 1 \) and \( |f(m)| \geq \delta(x, A) \). For \( m \in A \), we have

\[
\|m\| = \|f\||m|| \\
\geq |f(m)| \\
\leq \|x - m\|.
\]

Therefore \( m \perp_F x \) and \( A \perp_F x \).

**Theorem 2.8** Let \((X, \|\|)\) be a normed linear space and \( x \in X \).

(i) If a nonempty bounded set \( A \) in \( X \) is co-remotal then

\[ A \cap \bigcap_{g \in X} C_{\|x - a\|} \neq \emptyset, \]
where $C_{\|x-a\|} = A \cap B^c[g, \delta_2]$.

(ii) For every $x \in X$, if $A \cap (\bigcap_{g \in X} C_{\|x-g\|}) \neq \emptyset$. Then $A$ is co-remotal.

Proof. (i) Suppose $A$ is co-remotal and $x \in X$. Then there exists a $a_0 \in A$ such that $\|g - a_0\| \geq \|g - x\|$ for every $g \in A$. Therefore $a_0 \in C_{\|x-a\|}$ for every $g \in A$, it follows that $a_0 \in \bigcap_{g \in X} C_{\|x-g\|}$, and $A \cap (\bigcap_{g \in X} C_{\|x-g\|}) \neq \emptyset$.

(ii) Suppose $x \in X$, since $A \cap (\bigcap_{g \in X} C_{\|x-g\|}) \neq \emptyset$. There exists a $a_0 \in A$ such that $a_0 \in (\bigcap_{g \in X} C_{\|x-g\|})$. Therefore $\|a_0 - g\| \geq \|x - g\|$ for every $g \in A \backslash \{a_0\}$. Therefore $A$ is co-remotal.

Theorem 2.9 Let $(X, \|\cdot\|)$ be a normed linear space and $A$ a co-remotal subset of $X$, $A = A + A$ and $0 \in A$. If $C_A^{-1}(0)$ is singleton, then $A$ is co-uniquely remotal.

Proof. Suppose $x \in X$ and $a_1, a_2 \in C_A(x)$. Then $x \in C_{A^{-1}}(a_i)$ for $i = 1, 2$. Therefore $x - a_i \in C_{A^{-1}}(0)$ for $i = 1, 2$. It follow that $x - a_1 = x - a_2$ and $a_1 = a_2$. Thus $A$ is co-uniquely remotal.

Theorem 2.10 Let $(X, \|\cdot\|)$ be a normed linear space, and $A$ be a bounded subset. Then $C_A^{-1}(a_0)$ is convex.

Proof. If $x_1, x_2 \in C_A^{-1}(a_0)$ and $0 < \lambda < 1$. Since $\|a_0 - a\| \geq \|x_1 - a_0\|$ and $\|a_0 - a\| \geq \|x_2 - a_0\|$, for every $a \in A \backslash \{a_0\}$. Then

$$\|\lambda x_1 + (1 - \lambda)x_2 - a\| = \|\lambda(x_1 - a) + (1 - \lambda)(x_2 - a)\|$$

$$\leq \lambda \|x_1 - a\| + (1 - \lambda)\|x_2 - a\|$$

$$\leq \lambda \|a_0 - a\| + (1 - \lambda)\|a_0 - a\|,$$

for every $a \in A \backslash \{a_0\}$. Therefore $\lambda x_1 + (1 - \lambda)x_2 \in C_A^{-1}(a_0)$. It follows that $C_A^{-1}(a_0)$ is convex.

Theorem 2.11 Let $(X, \|\cdot\|)$ be a normed linear space, $A$ a subset of $X$, $-A = A$, $A = A + A$ and $0 \in A$. If $A$ is co-remotal, then $A$ is co-uniquely remotal.
**Proof.** Suppose \( x \in X \) and \( g_1, g_2 \in C_A(x) \) by \( g_1 \neq g_2 \). Since \( g_1, g_2 \in C_A(x) \), We have \( x + g_1, x + g_2 \in C_A^{-1}(0) \). Also \( -g_2 - x \in C_A^{-1}(0) \), therefore \( \frac{1}{2}[g_1 - g_2] = \frac{1}{2}[g_1 + x - x - g_2] \in C_A^{-1}(0) \). That is, for every \( a \in A \{0\} \),

\[
\| \frac{1}{2}[g_1 - g_2] - a \| \leq \|a\|.
\]

Since \( g_1 - g_2 \in A \) and \( a = (g_1 - g_2) \in A \). Then

\[
\| \frac{1}{2}[g_1 - g_2] + [g_1 - g_2] \| \leq \|g_1 - g_2\|
\]

and

\[
\frac{3}{2} \|g_1 - g_2\| \leq \|g_1 - g_2\|
\]

and

\[
\frac{3}{2} \leq 1
\]

is contraction. That is, \( A \) is co-uniquely remotal.

**Theorem 2.12** Let \((X, \|\cdot\|)\) be a normed linear space, \( A \) a subset of \( X \) and \( x \in X \). If \( A \) compact(weakly compact) then \( C_A(x) \) is compact(weakly compact).

**Proof.** Suppose \( \{x_n\}_{n \geq 1} \) is a sequence in \( C_A(x) \). Then for every sequence \( \{a_n\}_{n \geq 1} \) in \( A \{x\} \)

\[
\|x_n - a_n\| \geq \|x - a_n\|.
\]

Since \( A \) is compact, there exists a convergent subsequence \( \{a_{n_k}\} \) and \( \{x_{n_l}\} \) in \( A \), \( x_0 \) and \( a_0 \) \( \in A \) such that \( x_{n_l} \rightarrow x_0 \) and \( a_{n_k} \rightarrow a_0 \). Then \( \|x_{n_l} - a_{n_k}\| \geq \|x - a_{n_k}\| \). Then \( \|x_0 - a_0\| \geq \|x - a_0\| \). Therefore \( x_0 \in C_A(x) \) and \( x_{n_l} \rightarrow x_0 \). Therefore \( \{x_n\}_{n \geq 1} \) has a subsequence in \( C_A(x) \) and \( C_A(x) \) is compact (weakly compact).

**Theorem 2.13** Let \( A \) be a compact subset of a normed linear space \((X, \|\cdot\|)\). Then

(i) for every \( x \in X \), \( C_A(x) \),

(ii) \( C_A \) is upper semi-continues on \( D(C_A) \).

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Proof. (i) Suppose \( \{a_n\}_{n \geq 1} \) is any sequence in \( C_A(x) \). Therefore for every \( n \geq 1 \), \( \|a_n - a\| \geq \|x - a\| \) for every \( a \in A \setminus \{a_n\} \). Since \( A \) is compact, the sequence \( \|a_n\|_{n \geq 1} \) has a subsequence \( \{a_{n_i}\} \) such that \( a_{n_i} \to a_0 \in A \). Therefore

\[
\|a_0 - a\| = \lim_{i \to \infty} \|a_{n_i} - a\| \geq \|x - a\|,
\]

for every \( a \in A \setminus \{a_n\} \), it follows that \( a_0 \in C_A(x) \). Thus \( C_A(x) \) is compact.

(ii) Suppose \( N \) is a closed subset of \( A \) and \( B = \{x \in D(C_A) : C_A(x) \cap N \neq \emptyset\} \). To show that \( B \) is closed, if \( x \) is a limit point of \( B \). Then there exists a sequence \( \{x_n\}_{n \geq 1} \) in \( B \) such that \( x_n \to x \). Now, \( x_n \in B \), implies that there exists \( a_n \in C_A(x_n) \cap N \), and so \( \|a_n - a\| \geq \|x_n - a\| \) for every \( a \in A \setminus \{a_n\} \). Since \( A \) is compact, there exists a subsequence \( \{a_{n_i}\}_{i \geq 1} \) of \( \{a_n\}_{n \geq 1} \) such that \( a_{n_i} \to a_0 \), and so \( \|a_{n_i} - a\| \geq \|x_{n_i} - a\| \) for every \( a \in A \setminus \{a_{n_i}\} \). Implies that \( \|a_0 - a\| \geq \|x - a\| \) for every \( a \in A \setminus \{a_0\} \). Therefore \( a_0 \in C_A(x) \cap N \), i.e., \( x \in B \), so that \( B \) is closed. Therefore \( C_A \) is upper semi-continuous.

Theorem 2.14 Let \( A \) be a compact subset of a normed linear space \((X, \|\cdot\|)\). Then for every subset \( B \) of \( D(C_A) \), the subset \( C_A(B) \) is compact in \( A \).

Proof. Suppose \( \{a_n\}_{n \geq 1} \) is a sequence in \( C_A(B) \). Then there exists a \( x_n \in B \), such that \( a_n \in C_A(x_n) \), so that \( \|a_n - a\| \geq \|x_n - a\| \) for every \( a \in A \setminus \{a_n\} \). Since \( A \) is compact, there exists a subsequence \( \{a_{n_i}\}_{i \geq 1} \) of \( \{a_n\}_{n \geq 1} \) such that \( a_{n_i} \to a_0 \in A \). Since \( x_{n_i} \in A \), the compactness of \( B \) implies that the existence of a subsequence \( \{x_{n_{i_m}}\}_{m \geq 1} \) such that \( x_{n_{i_m}} \to x \in B \). Now, \( a_{n_{i_m}} \in C_A(x_{n_{i_m}}) \), implies \( \|a_{n_{i_m}} - a\| \geq \|x_{n_{i_m}} - a\| \) for every \( a \in A \setminus \{a_{n_{i_m}}\} \), in limiting case implies \( \|a_0 - a\| \geq \|x - a\| \) for every \( a \in A \setminus \{a_0\} \). Therefore \( a_0 \in C_A(x) \subseteq C_A(B) \). Hence \( C_A(B) \) is compact.

References


