On the strong convergence theorems by the hybrid method for a family of mappings in uniformly convex Banach spaces

M. Salehi\textsuperscript{a,1}, V. Dadashi \textsuperscript{b,2}, M. Roohi\textsuperscript{c,3},

\textsuperscript{a}Department of Mathematics, Islamic Azad University, Savadkooh Branch, Savadkooh, Iran.
\textsuperscript{b}Department of Mathematics, Islamic Azad University, Sari Branch, Sari, Iran.
\textsuperscript{c}Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran.

Received 22 November 2010; Accepted 4 April 2011

Abstract

Some algorithms for finding common fixed point of a family of mappings is constructed. Indeed, let C be a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and let \( \{T_n\} \) be a family of self-mappings on C such that the set of all common fixed points of \( \{T_n\} \) is nonempty. We construct a sequence \( \{x_n\} \) generated by the hybrid method and also we give the conditions of \( \{T_n\} \) under which \( \{x_n\} \) converges strongly to a common fixed point of \( \{T_n\} \).

Keywords: Hybrid method, Common fixed point, Iterative algorithm, Uniformly convex Banach space.

1 Introduction

Let \( \{T_n\}_{n=0}^{+\infty} \) be a family of mappings of a real Hilbert space \( \mathcal{H} \) into itself and let \( F(T_n) \) be the set of all fixed points of \( T_n \). By the assumption that \( \bigcap_{n=0}^{+\infty} F(T_n) \neq \emptyset \), Haugazeau

\textsuperscript{1}msalehi76@yahoo.com
\textsuperscript{2}vahid.dadashi@iausari.ac.ir
\textsuperscript{3}mehdi.roohi@gmail.com
M. Salehi, V. Dadashi, M. Roshi

[4] introduced a sequence \( \{x_n\} \) generated by the hybrid method, as following

\[
\begin{aligned}
x_0 &\in \mathcal{H} \\
y_n &= T_n(x_n) \\
C_n &= \{ z \in \mathcal{H} : \langle x_n - y_n, y_n - z \rangle \geq 0 \} \\
Q_n &= \{ z \in \mathcal{H} : \langle x_n - z, x_0 - x_n \rangle \geq 0 \} \\
x_{n+1} &= P_{C_n \cap Q_n}(x_0).
\end{aligned}
\]

In case that \( C_i \) is a closed convex subset of \( \mathcal{H} \) for \( i = 1, \ldots, m \), \( \bigcap_{i=1}^{m} C_i \neq \emptyset \) and \( T_n = P_{C_n(m \mod m + 1)} \), he proved a strong convergence theorem. Recently, Solodov and Svaiter [9], Bauschke and Combettes [2], Atsushiba and Takahashi [1], Nakajo and Takahashi [8], Iiduka, Takahashi and Toyoda [5], Nakajo, Shimoji and Takahashi [7], Svaiter [9], Bauschke and Combettes [2], Atsushiba and Takahashi [1], Nakajo and

Throughout this paper, let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and let \( X \) be a real Banach space with dual space \( X^* \). The line segment between \( x \) and \( y \) is denoted and defined by \( [x,y] := \{tx + (1-t)y : t \in [0,1]\} \). For a set-valued mapping \( T : X \rightrightarrows Y \), the domain of \( T \) is \( \text{Dom}(T) = \{ x \in X : T(x) \neq \emptyset \} \), range of \( T \) is \( \text{R}(T) = \{ y \in Y : \exists x \in X, (x,y) \in T \} \) and the inverse \( T^{-1} \) of \( T \) is \( \{ (y,x) : (x,y) \in T \} \). For a real number \( c \), let \( cT = \{ (xy) : (x,y) \in T \} \). If \( S \) and \( T \) are any set-valued mappings, we define \( S + T = \{ (xy + z) : (x,y) \in S, (x,z) \in T \} \). Set \( R_0^+ = [0, +\infty) \) and

\[
\mathcal{G} = \{ g : R_0^+ \rightarrow R_0^+ : g(0) = 0, \text{ g is continuous, strictly increasing and convex}\}.
\]

**Lemma 1.1.** [3] Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) and let \( x \in X \). Then, there exists a unique element \( x_0 \in C \) such that \( \|x_0 - x\| = \inf_{y \in C} \|y - x\| \). Putting \( x_0 = P_C(x) \), we call \( P_C \) the metric projection onto \( C \).

**Lemma 1.2.** [10] Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) whose norm is Gateaux differentiable and let \( x \in X \). Then \( y = P_C(x) \) if and only if \( \langle y - z, J(x - y) \rangle \geq 0 \) for all \( z \in C \).

**Lemma 1.3.** [10] Suppose \( X \) has a Gateaux differentiable norm. Then the duality mapping \( J \) is single-valued and \( \|x\|^2 - \|y\|^2 \geq 2\langle x - y, Jx \rangle \) for all \( x, y \in X \).

**Lemma 1.4.** [11] The Banach space \( X \) is uniformly convex if and only if for every bounded subset \( B \) of \( X \), there exists \( g_B \in \mathcal{G} \) such that

\[
\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g_B(\|x - y\|) \tag{1.2}
\]

for all \( x, y \in B \) and all \( \lambda \in [0,1] \).
2 Main results

Let \( \{T_n\}_{n=0}^{+\infty} \) be a family of self-mappings of \( C \) and \( F(T_n) \) be the set of all fixed points of \( T_n \). Assume that \( F := \bigcap_{n=0}^{+\infty} F(T_n) \) is a nonempty closed convex subset of \( C \) satisfies the following condition, \( \exists x_0 \in C \ \exists \{a_n\} \subseteq (0, +\infty) \) with \( \liminf \alpha_n > 0 \ \exists \{\alpha_n\} \subseteq [0, 1], \ \exists \{\beta_n\} \subseteq [0, 1] \) such that

\[
\langle x - z, J(x - w_n) \rangle \geq \alpha_n \|x - w_n\|^2
\]

(2.1)

for all \( x \in C, z \in F(T_n) \), where, \( w_n = \beta_n T_0(x_0) + (1 - \beta_n) T_n(a_n x_0 + (1 - a_n)x) \).

Algorithm 2.1. Let \( \{T_n\} \) be a family of self-mappings of \( C \) with \( F \neq \emptyset \) which satisfies condition (2.1). Let \( \{x_n\}_{n=1}^{+\infty} \) be a sequence generated by the following algorithm.

\[
\begin{align*}
x_0 & \in C, n \in \mathbb{N}_0 \\
y_n & = \alpha_n x_0 + (1 - \alpha_n)x_n \\
z_n & = \beta_n T_0(x_0) + (1 - \beta_n) T_n(y_n) \\
C_n & = \{z \in C : \langle x_n - z, J(x_n - z_n) \rangle \geq a_n \|x_n - z_n\|^2\} \\
Q_n & = \{z \in C : \langle x_n - z, J(x_0 - x_n) \rangle \geq 0\} \\
x_{n+1} & = P_{C_n \cap Q_n}(x_0)
\end{align*}
\]

(2.2)

Theorem 2.2. Suppose \( C \) is a nonempty closed convex subset of a uniformly convex Banach space \( X \) whose norm is Gateaux differentiable and \( \{T_n\} \) is a family of self-mappings of \( C \) with \( F \neq \emptyset \) which satisfies the condition (2.1). Assume that

\( (*) \) for every bounded sequence \( \{u_n\} \) in \( C \), \( \sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty \) and \( \sum_{n=0}^{+\infty} a u_{n+1} - u_n \| < +\infty \) for some \( g \in C \) and some \( u \in [T_0(x_0), T_n(w)] \), where \( w \in [x_0, u_n] \) and \( a > 0 \) imply that \( w_n(u_n) \subseteq F \). Then the sequence \( \{x_n\} \) generated by Algorithm 2.1 converges strongly to \( P_F(x_0) \).

Proof. We split the proof into six steps.

Step 1. \( \{x_n\} \) is well defined.

Notice that \( C_n \) and \( Q_n \) are closed and convex sets for all \( n \in \mathbb{N}_0 \). On the other hand, condition (2.1) and the definition of \( C_n \) in Algorithm 2.1 imply that \( F(T_n) \subseteq C_n \) for all \( n \in \mathbb{N}_0 \). Hence \( F \subseteq C_n \) for all \( n \in \mathbb{N}_0 \). Since \( J(0) = 0 \), it follows from the definition of \( Q_n \) in Algorithm 2.1 that \( Q_n = C \) which implies that \( F \subseteq C_0 \cap Q_0 \).

Lemma 1.1 guarantees that there exists a unique element \( x_1 = P_{C_0 \cap Q_0}(x_0) \). By Lemma 1.2,

\[
\langle x_1 - z, J(x_0 - x_1) \rangle \geq 0
\]

for all \( z \in C_0 \cap Q_0 \) and hence by \( F \subseteq C_0 \cap Q_0 \) we get

\[
\langle x_1 - z, J(x_0 - x_1) \rangle \geq 0
\]
for all $z \in F$. Therefore, $F \subseteq Q_1$ and so apply the fact that $F \subseteq C_n$ for all $n \in \mathbb{N}_0$ we have $F \subseteq C_1 \cap Q_1$. Again, Lemma 1.1 guarantees that there exists a unique element $x_2 = P_{C_1 \cap Q_1}(x_0)$. Inductively, we find that $\{x_n\}$ is well defined.

**Step 2.** $\{x_n\}$ is a bounded sequence.
From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and $F \subseteq C_n \cap Q_n$ for all $n \in \mathbb{N}_0$ we have

$$\|x_{n+1} - x_0\| \leq \|x_0 - P_F(x_0)\|$$

(2.3)

for all $n \in \mathbb{N}_0$, which implies that $\{x_n\}$ is a bounded sequence.

**Step 3.** $\lim_{n \to \infty} \|x_n - x_0\|$ exists.
Replace terms $x_{n+1} - x_0$ and $x_n - x_o$ respectively with $x$ and $y$ in Lemma 1.3,

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_0, J(x_n - x_0) \rangle$$

and hence $x_{n+1} \in Q_n$ implies that $\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2$ for all $n \in \mathbb{N}_0$; i.e., $\|x_n - x_0\|$ is an increasing sequence and so by Step 2 we find that $\lim_n \|x_n - x_o\|$ exists.

**Step 4.** $\sum_{n=0}^{\infty} g(\|x_{n+1} - x_n\|) < +\infty$ for some $g \in G$.
It follows from Lemma 1.4 that there exists $g \in G$ such that

$$\frac{x_n + x_{n+1}}{2} - x_0 \|^2 \leq \frac{1}{4} \|x_n - x_0\|^2 + \frac{1}{2} \|x_{n+1} - x_0\|^2 - \frac{1}{4} g(\|x_{n+1} - x_n\|)$$

and hence

$$g(\|x_{n+1} - x_n\|) \leq 2\|x_n - x_0\|^2 + 2\|x_{n+1} - x_0\|^2 - 4\|\frac{x_n + x_{n+1}}{2} - x_0\|^2$$

(2.4)

for all $n \in \mathbb{N}_0$. From Lemma 1.2 and the definition of $Q_n$ we get $x_n = P_{Q_n}(x_0)$ and so by $x_{n+1} \in Q_n$ and convexity of $Q_n$ we get $\frac{x_n + x_{n+1}}{2} \in Q_n$. Again, by $x_n = P_{Q_n}(x_0)$,

$$\|\frac{x_n + x_{n+1}}{2} - x_0\|^2 \geq \|x_n - x_0\|^2.$$  

(2.5)

It follows from inequalities (2.4) and (2.5) that

$$g(\|x_{n+1} - x_n\|) \leq 2\|x_{n+1} - x_0\|^2 - 2\|x_n - x_0\|^2 \text{ for all } n \in \mathbb{N}_0.$$  

(2.6)

That $\sum_{n=0}^{\infty} g(\|x_{n+1} - x_n\|) < +\infty$ follows from (2.6) and Step 3.

**Step 5.** $\sum_{n=0}^{\infty} g(a\|x_n - z_n\|) < +\infty$ for some $g \in G$ and $a > 0$.
Since $a_n > 0$ for all $n \in \mathbb{N}_0$ and $\lim inf a_n > 0$, there exists $a > 0$ for which $a_n \geq a$ for all $n \in \mathbb{N}_0$. Now, $x_{n+1} \in C_n$ guarantees that

$$\|x_n - x_{n+1}\| \|x_n - z_n\| \geq \langle x_n - x_{n+1}, J(x_n - z_n) \rangle \geq a_n \|x_n - z_n\|^2$$
and thus
\[ a \| x_n - z_n \| \leq \| x_{n+1} - x_n \| \] (2.7)
for all \( n \in \mathbb{N}_0 \). That \( \sum_{n=0}^{+\infty} g(a \| x_n - z_n \|) < +\infty \) follows from (2.7), (1.1) and Step 4.

**Step 6.** \( \{ x_n \} \to P_F(x_0) \).

It follows from our assumption, Step 4 and Step 5 that \( w_n(x_n) \subseteq F \). Let the subsequence \( \{ x_n \} \) of \( \{ x_n \} \) converges weakly to \( w \in F \). Therefore, weakly lower semicontinuity of the norm and (2.3) imply that
\[ \| P_F(x_0) - x_0 \| \leq \| w - x_0 \| \leq \lim_{i \to +\infty} \| x_n - x_0 \| \leq \| P_F(x_0) - x_0 \| \]
and hence \( x_n \to w = P_F(x_0) \).

**Corollary 2.3.** Suppose \( C \) is a nonempty closed convex subset of a uniformly convex Banach space \( X \) whose norm is Gateaux differentiable and \( \{ T_n \} \) is a family of self-mappings of \( C \) with \( F \neq \emptyset \) which satisfies the following condition.

(a) \( \exists x_0 \in C \) \( \exists \{ a_n \} \subseteq (0, +\infty) \) with \( \lim_{n} a_n > 0 \) \( \exists \{ \alpha_n \} \subseteq [0, 1] \) such that
\[ \langle x - z, J(x - T_n(v_n)) \rangle \geq a_n \| x - T_n(v_n) \|^2 \]
for all \( x \in C \), \( z \in F(T_n) \), where, \( v_n = \alpha_n x_0 + (1 - \alpha_n)x \);

(b) for every bounded sequence \( \{ u_n \} \in C \), \( \sum_{n=0}^{+\infty} g(\| u_{n+1} - u_n \|) < +\infty \) and \( \sum_{n=0}^{+\infty} g(a \| u_n - u \|) < +\infty \) for some \( g \in G \) and some \( u \in [T_0(x_0), T_n(w)] \), where \( w \in [x_0, u_n] \) and \( a > 0 \) imply that \( w_n(u_n) \subseteq F \).

Then the sequence \( \{ x_n \} \) generated by the following algorithm converges strongly to \( P_F(x_0) \).

\[
\begin{align*}
\{ \ & n \in \mathbb{N}_0, \\
& y_n = \alpha_n x_0 + (1 - \alpha_n)x_n \\
& z_n = T_n(y_n) \\
& C_n = \{ z \in C : \langle x_n - z, J(x_n - z_n) \rangle \geq a_n \| x_n - z_n \|^2 \} \\
& Q_n = \{ z \in C : \langle x_n - z, J(x_n - x_n) \rangle \geq 0 \} \\
& x_{n+1} = P_{C_n \cap Q_n}(x_0) \} \end{align*} \]
(2.8)

**Proof.** All conditions of Theorem 2.2 hold for \( \beta_n = 0 \) and also in this case (2.2) reduces to (2.8). So Theorem 2.2 implies the result.

**Corollary 2.4.** Suppose \( C \) is a nonempty closed convex subset of a uniformly convex Banach space \( X \) whose norm is Gateaux differentiable and \( \{ T_n \} \) is a family of self-mappings of \( C \) with \( F \neq \emptyset \) which satisfies the following condition.

(a) \( \exists x_0 \in C \) \( \exists \{ a_n \} \subseteq (0, +\infty) \) with \( \lim_{n} a_n > 0 \) \( \exists \{ \beta_n \} \subseteq [0, 1] \)
\[ \langle x - z, J(x - w_n) \rangle \geq a_n \| x - w_n \|^2 \]
for all $x \in C$, $z \in F(T_n)$, where, $w_n = \beta_nT_0(x_0) + (1 - \beta_n)T_n(x)$;

(b) for every bounded sequence $\{u_n\}$ in $C$, $\sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ and $\sum_{n=0}^{+\infty} g(a\|u_n - T_n(u_n)\|) < +\infty$ for some $g \in G$, $w_n = \beta_nT_0(x_0) + (1 - \beta_n)T_n(u_n)$, and $a > 0$ imply that $w_n(u_n) \subseteq F$.

Then $\{x_n\}$ generated by the following algorithm converges strongly to $P_F(x_0)$.

$$
\begin{align*}
x_0 & \in C, n \in \mathbb{N}_0 \\
z_n & = \beta_nT_0(x_0) + (1 - \beta_n)T_n(x_n) \\
C_n & = \{z \in C : \langle x_n - z, J(x_n - z) \rangle \geq a_n\|x_n - z\|^2\} \\
Q_n & = \{z \in C : \langle x_n - z, J(x_0 - x_n) \rangle \geq 0\} \\
x_{n+1} & = P_{C_n \cap Q_n}(x_0)
\end{align*}
$$

(2.9)

**Proof.** Similar to Corollary 2.3, all conditions of Theorem 2.2 hold for $\alpha_n = 0$ and so with this assumption, (2.2) collapses to (2.9) which it completes the proof.

**Corollary 2.5.** Suppose $C$ is a nonempty closed convex subset of a uniformly convex Banach space $X$ whose norm is Gateaux differentiable and $\{T_n\}$ is a family of self-mappings of $C$ with $F \neq \emptyset$ which satisfies the following conditions.

(a) $\exists\{a_n\} \subseteq (0, +\infty)$ with $\liminf_n a_n > 0$

$$
\langle x - z, J(x - T_n(x)) \rangle \geq a_n\|x - T_n(x)\|^2
$$

for all $x \in C$, $z \in F(T_n)$;

(b) for every bounded sequence $\{u_n\}$ in $C$, $\sum_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ and $\sum_{n=0}^{+\infty} g(a\|u_n - T_n(u_n)\|) < +\infty$ for some $g \in G$ and $a > 0$ imply that $w_n(u_n) \subseteq F$.

Then $\{x_n\}$ generated by the following algorithm converges strongly to $P_F(x_0)$.

$$
\begin{align*}
x_0 & \in C, n \in \mathbb{N}_0 \\
C_n & = \{z \in C : \langle x_n - z, J(x_n - T_n(x_n)) \rangle \geq a_n\|x_n - T_n(x_n)\|^2\} \\
Q_n & = \{z \in C : \langle x_n - z, J(x_0 - x_n) \rangle \geq 0\} \\
x_{n+1} & = P_{C_n \cap Q_n}(x_0)
\end{align*}
$$

**Proof.** Put $\alpha_n = \beta_n = 0$ in Theorem 2.2.

**Corollary 2.6.** Suppose $C$ is a nonempty closed convex subset of a real Hilbert space $H$ and $\{T_n\}$ is a family of self-mappings of $C$ with $F \neq \emptyset$ which satisfies the following conditions.

(a) $\exists x_0 \in C \\exists\{b_n\} \subseteq (-1, +\infty)$ with $\liminf_n b_n > -1$ and $\exists\{\alpha_n\} \subseteq [0, 1]$, $\exists\{\beta_n\} \subseteq [0, 1]$ such that

$$
\|w_n - z\|^2 \leq \|x - z\|^2 - b_n\|x - w_n\|^2
$$

for all $x \in C$, $z \in F(T_n)$, where, $v_n = \alpha_n x_0 + (1 - \alpha_n)x$ and $w_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(v_n)$;
(b) for every bounded sequence \( \{u_n\} \) in \( C \), \( \sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty \) and \( \sum_{n=0}^{+\infty} (a\|u_n - q_n\|^2) < +\infty \), where \( q_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(p_n) \), \( p_n = \alpha_n x_0 + (1 - \alpha_n)u_n \) and \( a > 0 \) imply that \( w_n(u_n) \subseteq F \).

Then \( \{x_n\} \) generated by the following algorithm converges strongly to \( P_F(x_0) \).

\[
\begin{align*}
y_n &= \alpha_n x_0 + (1 - \alpha_n)x_n \\
z_0 &= T_0(x_0) \\
z_n &= \beta_n z_0 + (1 - \beta_n)T_n(y_n) \quad (n \geq 1) \\
C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n\|x_n - z_n\|^2\} \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n}(x_0).
\end{align*}
\tag{2.10}
\]

**Proof.** First we note that, for \( x \in C, z \in F(T_n), v_n = \alpha_n x_0 + (1 - \alpha_n)x \) and \( w_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(v_n) \), by our assumption we have \( \|w_n - z\|^2 \leq \|x - z\|^2 - b_n\|x - w_n\|^2 \) for all \( z \in F(T_n) \), if and only if

\[
\|w_n - x\|^2 + 2\langle w_n - x, x - z \rangle + \|x - z\|^2 \leq \|x - z\|^2 - b_n\|x - w_n\|^2
\]

if and only if \( \langle x - z, x - w_n \rangle \geq \frac{1 + b_n}{2}\|x - w_n\|^2 \). Then condition (2.1) satisfies for \( a_n = \frac{1 + b_n}{2} \). In a real Hilbert space \( H \), we have

\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2
\]

for all \( x, y \in H \) and \( \lambda \in [0, 1] \), so, we can consider \( g_B(t) = t^2 \) for each bounded subset \( B \) of \( H \) in Lemma 1.4 and hence (*) holds. Then all assumptions of Theorem 2.2 hold which it implies that \( \{x_n\} \) converges strongly to \( P_F(x_0) \).

By putting \( \beta_n = 0, \alpha_n = 0 \) and \( \alpha_n = \beta_n = 0 \) in (2.10) we get the following results respectively.

**Corollary 2.7.** Suppose \( C \) is a nonempty closed convex subset of a real Hilbert space \( H \) and \( \{T_n\} \) is a family of self-mappings of \( C \) with \( F \neq \emptyset \) which satisfies the following conditions.

(a) \( \exists x_0 \in C \) \( \exists \{b_n\} \subseteq (-1, +\infty) \) with \( \liminf \limits_{n \to \infty} b_n > -1 \) and \( \exists \{\alpha_n\} \subseteq [0, 1] \) such that

\[
\|T_n(v_n) - z\|^2 \leq \|x - z\|^2 - b_n\|x - T_n(v_n)\|^2
\]

for all \( x \in C, z \in F(T_n) \), where, \( v_n = \alpha_n x_0 + (1 - \alpha_n)x \);

(b) for every bounded sequence \( \{u_n\} \) in \( C \), \( \sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty \) and \( \sum_{n=0}^{+\infty} (a\|u_n - T_n(v_n)\|^2) < +\infty \), where \( v_n = \alpha_n x_0 + (1 - \alpha_n)u_n \) and \( a > 0 \) imply that \( w_n(u_n) \subseteq F \).

Then \( \{x_n\} \) generated by the following algorithm converges strongly to \( P_F(x_0) \).

\[
\begin{align*}
y_n &= \alpha_n x_0 + (1 - \alpha_n)x_n \\
z_n &= T_n(y_n) \\
C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n\|x_n - z_n\|^2\} \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n}(x_0).
\end{align*}
\]
Corollary 2.8. Suppose $C$ is a nonempty closed convex subset of a real Hilbert space $H$ and $\{T_n\}$ is a family of self-mappings of $C$ with $F \neq \emptyset$ which satisfies the following conditions.

(a) $\exists x_0 \in C \exists \{b_n\} \subseteq (-1, +\infty)$ with $\liminf_n b_n > -1$ and $\exists \{\beta_n\} \subseteq [0, 1]$ such that $\|w_n - z\|^2 \leq \|x - z\|^2 - b_n \|x - w_n\|^2$

for all $x \in C$, $z \in F(T_n)$, where $w_n = \beta_n T_0(x_0) + (1 - \beta_n) T_n(x)$;

(b) for every bounded sequence $\{u_n\}$ in $C$, $\sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty$ and $\sum_{n=0}^{+\infty} (a\|u_n - w_n\|)^2 < +\infty$, where $w_n = \beta_n T_0(x_0) + (1 - \beta_n) T_n(u_n)$ and $a > 0$ imply that $w_n(u_n) \subseteq F$.

Then $\{x_n\}$ generated by the following algorithm converges strongly to $P_F(x_0)$.

\[
\begin{align*}
z_0 &= T_0(x_0) \\
z_n &= \beta_n z_0 + (1 - \beta_n) T_n(x_n) \ (n \geq 1) \\
C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n \|x_n - z_n\|^2\} \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n}(x_0).
\end{align*}
\]

Corollary 2.9. [6] Suppose $C$ is a nonempty closed convex subset of a real Hilbert space $H$ and $\{T_n\}$ is a family of self-mappings of $C$ with $F \neq \emptyset$ which satisfies the following conditions.

(a) $\exists \{b_n\} \subseteq (-1, +\infty)$ with $\liminf_n b_n > -1$ such that $\|T_n(x) - z\|^2 \leq \|x - z\|^2 - b_n \|x - T_n(x)\|^2$

for all $x \in C$, $z \in F(T_n)$;

(b) for every bounded sequence $\{u_n\}$ in $C$, $\sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty$ and $\sum_{n=0}^{+\infty} \|u_n - T_n u_n\|^2 < +\infty$ imply that $w_n(u_n) \subseteq F$.

Then $\{x_n\}$ generated by the following algorithm converges strongly to $P_F(x_0)$.

\[
\begin{align*}
x_0 &= C \\
z_n &= T_n(x_n) \\
C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n \|x_n - z_n\|^2\} \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n}(x_0).
\end{align*}
\]

3 Acknowledgment

Masood Salehi is supported by the Islamic Azad University–Savakouh Branch and Vahid Dadashi is supported by the Islamic Azad University–Sari Branch.
References


