

## On The Perimeter of an Ellipse

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### Abstract

Let  $E$  be the ellipse with major and minor radii  $a$  and  $b$  respectively, and  $P$  be its perimeter, then

$$P = \lim_{n \rightarrow \infty} 4 \tan \frac{\pi}{n} \left( a + b + 2 \sum_{k=2}^{2^{m-2}} \sqrt{a^2 \cos^2 \frac{(2k-2)\pi}{n} + b^2 \sin^2 \frac{(2k-2)\pi}{n}} \right),$$

where  $n = 2^m$ . So without considering the limit, it gives a reasonable approximation for  $P$ , it means that we can choose  $n$  large enough such that the amount of error be less than any given small number. On the other hand, the formula satisfies both limit status  $b \rightarrow a$  and  $b \rightarrow 0$  which give respectively  $P = 2\pi a$  and  $P = 4a$ .

**Keywords:** Ellipse, Perimeter, Surrounding polygon.

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## 1 introduction

Let  $E$  be the ellipse with cartesian equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . If  $P(E)$  refers to the perimeter of  $E$ , then we have

$$P(E) = 4 \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

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Unfortunately this integral dose not have an analytic solution, and so there are many approximations  $f(a, b)$  for  $P(E)$ , for example *Ramanujan II*, and *Cantrell* which are as follows (one can find many such approximations in [1])

$$R(a, b) = \pi(a + b) \left( 1 + \frac{3h}{10 + \sqrt{4 - 3h}} \right), \quad C(a, b) = 4(a + b) - 2(4 - \pi) \frac{ab}{H_p}$$

where

$$h = \frac{(a - b)^2}{(a + b)^2}, \quad H_p = \left( \frac{a^p + b^p}{2} \right)^{\frac{1}{p}}$$

in this article we present an approximation based on some surrounding polygons which are the images of surrounding regular polygons of a circle  $C$  under a transformation  $S$  which maps  $C$  onto  $E$ . If  $P_n$  be the perimeter of a surrounding  $n$ -gon of  $E$ , we will see that  $P_n \rightarrow P(E)$  as  $n \rightarrow \infty$ , hence  $|P_n - P(E)| \rightarrow 0$ . Meanwhile if we put  $\epsilon = |f(a, b) - P(E)|$ , there exist a natural number  $n_\epsilon$  such that  $|P_{n_\epsilon} - P(E)| < \epsilon$ .

Maybe the oldest approximation be  $f(a, b) = \pi(a + b)$  which is the average of  $2\pi a$  and  $2\pi b$ , on the other hand the renowned lower bound for  $P(E)$  is  $2\pi\sqrt{ab}$ . Clearly we can use *surrounded* polygons instead of surrounding polygons, and then we will obtain an approximation  $Q_n$  such that  $Q_n \rightarrow P(E)$  as  $n \rightarrow \infty$  and we have  $Q_n < P(E) < P_n$ , hence for each  $n$ ,  $Q_n$  and  $P_n$  are ,respectively, lower and upper bounds for  $P(E)$ .

## 2 surrounding polygon approximation

Consider the circle  $C$  with radius  $b$ , and its surrounding polygons. We will compute the coordinates of the corners using some trigonometry, if  $A(x_1, y_1)$  and  $B(x_2, y_2)$  be two adjacent corners, then  $S(A)(\frac{a}{b}x_1, y_1)$  and  $S(B)(\frac{a}{b}x_2, y_2)$  are two adjacent corners of a surrounding polygon of the ellipse  $E = S(C)$ , where  $S : R^2 \rightarrow R^2$  is a map defined by  $S(x, y) = (\frac{a}{b}x, y)$ .

**Lemma 2.1.** *The image of the circle  $C$  under the map  $S$  described above, is the ellipse  $E$ , moreover if  $AB$  is a tangent segment to  $C$  at the point  $T$ , then  $S(A)S(B)$  is a tangent segment to  $E$  at the point  $S(T)$ .*

*Proof.* The equation of  $C$  is  $x^2 + y^2 = b^2$ , we have to show that the point  $S(A)(\frac{a}{b}x, y)$  satisfies the equation of  $E$ , to this end we have  $\frac{(\frac{a}{b}x)^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{b^2} + \frac{y^2}{b^2} = \frac{1}{b^2}(x^2 + y^2) = \frac{b^2}{b^2} = 1$ . If  $A(x_1, y_1)$ ,  $B(x_2, y_2)$  and  $T(\alpha, \beta)$  then we know that  $y'(\alpha) = \frac{y_2 - y_1}{x_2 - x_1}$ , we have to show that  $Y'(\frac{a}{b}\alpha) = \frac{y_2 - y_1}{\frac{a}{b}(x_2 - x_1)}$ , but this is true since  $Y(\frac{a}{b}\alpha) = y(\alpha)$ , therefore  $\frac{a}{b}Y'(\frac{a}{b}\alpha) = y'(\alpha)$ . Here  $y$  and  $Y$  are extracted from the equations of  $C$  and  $E$ , respectively.  $\square$

Now consider the surrounding regular  $n$ -gon of  $C$ . For simplicity we consider the first quarter of  $C$ , we also assume that  $C$  is centered at origin. We start with the surrounding square and we make an octagon by drawing four tangent lines passing through the points which are the intersections of  $C$  and the segments connecting the center of  $C$  to the corners of the square. Each tangent line intersects two adjacent edges of the square and makes two adjacent corners of the surrounding regular octagon of  $C$  (Fig.1). We will continue this procedure to make any surrounding regular  $n$ -gon of  $C$  such that  $n = 2^m$ . Put  $N(0, b)$ ,  $M(b, 0)$  and Let  $A_1, A_2, \dots, A_p$  where  $p = 2^{m-2}$ , be the corners of the surrounding regular  $n$ -gon of  $C$  relevant to its first quarter. The following lemma will help us not only to find  $P_n$ , but also to compute the coordinates of  $A_k$ ,  $1 \leq k \leq p$ .

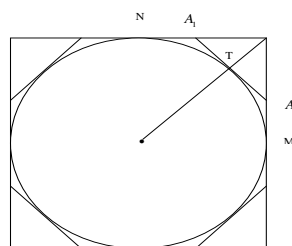


Fig. 1

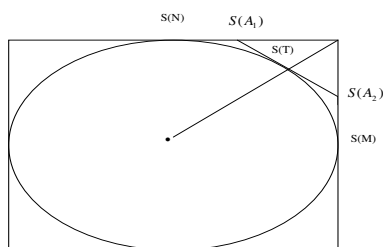


Fig. 2

**Lemma 2.2.** *If  $A_k(x_k, y_k)$  and  $A_{k+1}(x_{k+1}, y_{k+1})$  be two adjacent corners of the*

surrounding regular  $n$ -gon of  $C$ , then for  $1 \leq k \leq p-1$  we have

$$x_{k+1} = x_k + 2b \tan \frac{\pi}{n} \cos \frac{2k\pi}{n}, \quad y_{k+1} = y_k - 2b \tan \frac{\pi}{n} \sin \frac{2k\pi}{n}$$

*Proof.* For  $1 \leq k \leq p-1$ ,  $A_{k+1}$  is obtained by rotating the point  $A_1$  by  $\frac{2k\pi}{n}$  clockwise, hence

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \cos \frac{2k\pi}{n} & \sin \frac{2k\pi}{n} \\ -\sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

so

$$x_{k+1} = x_1 \cos \frac{2k\pi}{n} + y_1 \sin \frac{2k\pi}{n}, \quad y_{k+1} = -x_1 \sin \frac{2k\pi}{n} + y_1 \cos \frac{2k\pi}{n}$$

The similar equations can be written for  $x_k$  and  $y_k$ , then we compute  $x_{k+1} - x_k$ , and  $y_{k+1} - y_k$ , finally by using some trigonometry and keeping in mind that  $x_1 = b \tan \frac{\pi}{n}$ ,  $y_1 = b$  are the coordinates of  $A_1$ , the proof will be completed.  $\square$

Now  $S(A_k)$ ,  $1 \leq k \leq p$ , are the corners of a surrounding  $n$ -gon of  $E$  and we have

$$S(N)(0, b), \quad S(M)\left(\frac{a}{b}, 0\right) = (a, 0), \quad \text{and} \quad S(A_k)\left(\frac{a}{b}x_k, y_k\right) \quad 1 \leq k \leq p$$

Note that we express the length of the segment  $AB$  again by  $AB$ , meanwhile it is easy to verify that if  $AB$  is a horizontal segment, then  $S(A)S(B) = \left(\frac{a}{b}\right)AB$  and for a vertical segment  $AB$ ,  $S(A)S(B) = AB$ , therefore we have

$$S(N)S(A_1) = \frac{a}{b}b \tan \frac{\pi}{n} = a \tan \frac{\pi}{n}, \quad S(A_p)S(M) = b \tan \frac{\pi}{n}$$

$$S(A_k)S(A_{k+1}) = \sqrt{\left(\frac{a}{b}(x_{k+1} - x_k)\right)^2 + (y_{k+1} - y_k)^2} =$$

$$\sqrt{\left(2a \tan \frac{\pi}{n} \cos \frac{2k\pi}{n}\right)^2 + \left(2b \tan \frac{\pi}{n} \sin \frac{2k\pi}{n}\right)^2} = 2 \tan \frac{\pi}{n} \sqrt{a^2 \cos^2 \frac{2k\pi}{n} + b^2 \sin^2 \frac{2k\pi}{n}}$$

so, the perimeter of the surrounding  $n$ -gon of  $E$  is

$$\begin{aligned} P_n &= 4(S(N)S(A_1) + S(A_{2^{m-2}})S(M) + \sum_{k=1}^{2^{m-2}-1} S(A_k)S(A_{k+1})) = \\ &= 4(S(N)S(A_1) + S(A_{2^{m-2}})S(M) + \sum_{k=2}^{2^{m-2}} S(A_{k-1})S(A_k)) = \\ &= 4\left(a \tan \frac{\pi}{n} + b \tan \frac{\pi}{n} + 2 \tan \frac{\pi}{n} \sum_{k=2}^{2^{m-2}} \sqrt{a^2 \cos^2 \frac{(2k-2)\pi}{n} + b^2 \sin^2 \frac{(2k-2)\pi}{n}}\right) = \\ &= 4 \tan \frac{\pi}{n} \left(a + b + 2 \sum_{k=2}^{2^{m-2}} \sqrt{a^2 \cos^2 \frac{(2k-2)\pi}{n} + b^2 \sin^2 \frac{(2k-2)\pi}{n}}\right) \end{aligned}$$

**Theorem 2.3.** *If  $P = \lim_{n \rightarrow \infty} P_n$ , then*

- i.  $P = P(E)$ , the perimeter of  $E$*
- ii.  $\lim_{b \rightarrow a} P = 2\pi a$*
- iii.  $\lim_{b \rightarrow 0} P = 4a$*

*Proof.* i. First of all, note that for a fixed pair  $(a,b)$ ,  $P_n$  is decreasing (proof by triangle inequality; see Fig.2 to compare  $P_4$  and  $P_8$ ) and bounded below ( $P_n \geq P(E)$ ), therefore it is convergent. Now we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n &= \lim_{n \rightarrow \infty} 4(a+b) \tan \frac{\pi}{n} + \\ &\lim_{n \rightarrow \infty} 8 \tan \frac{\pi}{n} \sum_{k=2}^{\frac{n}{4}} \sqrt{a^2 \cos^2 \frac{(2k-2)\pi}{n} + b^2 \sin^2 \frac{(2k-2)\pi}{n}} = \\ &4 \lim_{n \rightarrow \infty} \frac{2\pi}{n} \left( \sum_{k=1}^{\frac{n}{4}} \sqrt{a^2 \cos^2 \frac{(2k-2)\pi}{n} + b^2 \sin^2 \frac{(2k-2)\pi}{n}} - a \right) = \\ &4 \lim_{n \rightarrow \infty} \frac{\pi}{4} \sum_{k=1}^{\frac{n}{4}} \sqrt{a^2 \cos^2 \frac{(k-1)\frac{\pi}{2}}{\frac{n}{4}} + b^2 \sin^2 \frac{(k-1)\frac{\pi}{2}}{\frac{n}{4}}} = \\ &4 \int_0^{\frac{\pi}{2}} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta = \\ &4 \int_0^a \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}} \, dx = 4 \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = P(E) \end{aligned}$$

Here we have used the change of variable  $x = a \sin \theta$

ii. Consider  $P_n$  and  $P$  as functions of  $b$ , defined on  $[0, a]$ , clearly  $P_n$  is continuous since it is a finite sum of continuous functions. To prove the continuity of  $P$ , we refer the reader to [2, theorem 2.27] with  $g(x) = \frac{a}{\sqrt{a^2 - x^2}}$ . Now we have

- 1.  $[0, a]$  is compact,
- 2.  $P_n \rightarrow P$ , pointwise on  $[0, a]$ ,
- 3.  $P_n(b) \geq P_{n+1}(b)$  for all  $b \in [0, a]$ ,  $n = 1, 2, 3, \dots$ .

(Note that by an abuse of language,  $P_{n+1}$  may be considered as  $P_{2n}$ , in other words we can formally define  $P_{n+i} = P_n$  for  $1 \leq i \leq n-1$ ). Hence, we conclude that  $P_n \rightarrow P$  uniformly on  $[0, a]$ , [3, theorem 7.13], so we are allowed to write

$$\lim_{b \rightarrow a} P = \lim_{b \rightarrow a} \lim_{n \rightarrow \infty} P_n(b) = \lim_{n \rightarrow \infty} \lim_{b \rightarrow a} P_n(b) = \lim_{n \rightarrow \infty} 4 \tan \frac{\pi}{n} (2^{n-1} a) =$$

$$\lim_{n \rightarrow \infty} 4 \tan \frac{\pi}{n} \left( \frac{n}{2} a \right) = 2\pi a$$

$$\begin{aligned} \text{iii. } \lim_{b \rightarrow 0} P &= \lim_{b \rightarrow 0} \lim_{n \rightarrow \infty} P_n(b) = \lim_{n \rightarrow \infty} 4 \tan \frac{\pi}{n} \left( a + 2 \sum_{k=2}^{\frac{n}{4}} a \cos \frac{2(k-1)\pi}{n} \right) = \\ &= \lim_{n \rightarrow \infty} 4 \tan \frac{\pi}{n} \left( -a + 2 \sum_{k=1}^{\frac{n}{4}} a \cos \frac{2(k-1)\pi}{n} \right) = \lim_{n \rightarrow \infty} \left( -4a \tan \frac{\pi}{n} \right) + \\ &= \lim_{n \rightarrow \infty} 8 \tan \frac{\pi}{n} \sum_{k=1}^{\frac{n}{4}} a \cos \frac{2(k-1)\pi}{n} = \lim_{n \rightarrow \infty} 8 \frac{\pi}{n} \sum_{k=1}^{\frac{n}{4}} a \cos \frac{2(k-1)\pi}{n} = \\ &= \lim_{n \rightarrow \infty} 4a \left( \frac{\pi}{\frac{n}{4}} \sum_{k=1}^{\frac{n}{4}} \cos \frac{(k-1)\frac{\pi}{2}}{\frac{n}{4}} \right) = 4a \int_0^{\frac{\pi}{2}} \cos x dx = 4a \end{aligned}$$

Note that we have used the fact that  $\lim_{n \rightarrow \infty} \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}} = 1$ .

□

We can call  $P_n$  an *infinite-type* approximation because it has a limit process, so we may call a formula without limit process, a *finite-type* approximation. Since  $P_n \rightarrow P(E)$ , for any given  $\epsilon > 0$  there exist a natural number  $N$  such that if  $n > N$ , then  $|P_n - P| < \epsilon$ , in other words choosing  $n$  large enough one may obtain a good approximation to evaluate the perimeter of  $E$ . Let  $f(a, b)$  be any finite-type approximation for  $P(E)$ , there is a number  $n$  such that  $|P_n - P| < |f(a, b) - P| = \epsilon$ .

## References

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