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Artinianess of Graded Generalized Local Cohomology Modules

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Abstract

Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous graded ring with local base ring $(R_0, \mathfrak{m}_0)$ of dimension d . Let $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ denote the irrelevant ideal of R and let M and N be two finitely generated graded R-modules. Let $t = t_{R_+}(M, N)$ be the first integer i such that $H^i_{R_+}(M, N)$ is not minimax. We prove that if $i \leq t$, then the set $Ass_{R_0}(H^i_{R_+}(M, N)_n)$ is asymptotically stable for $n \longrightarrow -\infty$ and $H^j_{\mathfrak{m}_0}(H^i_{R_+}(M, N))$ is Artinian for $0 \leq j \leq 1$. Moreover, let $s = s_{R_+}(M, N)$ be the largest integer i such that $H^i_{R_+}(M, N)$ is not minimax. For each $i \geq s$, we prove that $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M, N)$ is Artinian and that $H^j_{\mathfrak{m}_0}(H^i_{R_+}(M, N))$ is Artinian for $d-1 \leq j \leq d$. Finally we show that $H^{d-2}_{\mathfrak{m}_0}(H^s_{R_+}(M, N))$ is Artinian if and only if $H^d_{\mathfrak{m}_0}(H^{s-1}_{R_+}(M, N))$ is Artinian.

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1 Introduction

In this note $\mathbb{Z}$ denotes the set of all integer numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous graded

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ring with local base ring $(R_0, \mathfrak{m}_0)$ of dimension d . Let $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ denote the irrelevant ideal of R and let M and N be two finitely generated graded R -modules. For an integer $i \geq 0$, the i-th generalized local cohomology module of M and N with respect to R_+ is defined by

$$H^i_{R_+}(M,N) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(\frac{M}{{R_+}^n M}, N)$$

It is clear that if $M = R$, then $H^i_{R_+}(M,N)$ is the ordinary local cohomology module $H^i_{R_+}(N)$ of N with respect to R_+. By using [3,12.3.1], $H^i_{R_+}(M,N)$ has natural grading and its graded components are finitely generated R_0-modules [5]. So, Artinianess of graded generalized local cohomology allows us to draw conclusions on the multiplicity $e_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)$ of $H^i_{R_+}(M,N)$. Hence, one of the basic problem concerning generalized local cohomology is to finding when $H^j_{\mathfrak{m}_0}(H^i_{R_+}(M,N))$ is Artinian?

In this direction, we introduce two following important sets.

$$\Delta_1 = \{l \in \mathbb{Z} \mid H^j_{\mathfrak{m}_0}(H^i_{R_+}(M,N)) \ is\ Artinian\ for\ all\ i \leq l\ and\ 0 \leq j \leq 1\}$$

$$\Delta_2 = \{l \in \mathbb{Z} \mid H^j_{\mathfrak{m}_0}(H^i_{R_+}(M,N)) \ is\ Artinian\ for\ all\ i \geq l\ and\ d-1 \leq j \leq d\}$$

In recent years there have been several results showing that under certain conditions $H^j_{\mathfrak{m}_0}(H^i_{R_+}(M,N))$ is Artinian. For example, by [5], if $dim(R_0) \leq 1$, then $\Delta_1 = \Delta_2 = \mathbb{Z}$. Also, if R_+ is principal, then $\Delta_1 = \Delta_2 = \mathbb{Z}$, in view of [8, 2.5].

Let $c = cd_{R_+}(M,N) = sup\{i \geq 0 | H^i_{R_+}(M,N) \neq 0\}$ be the cohomological dimension of M and N with respect to R_+ , then $c \in \Delta_2$, by [8, 2.8]. Moreover, Sazeedeh in [7] showed that $H^1_{\mathfrak{m}_0}(H^1_{R_+}(N))$ is Artinian.

The specific statements, mentioned above, can be presented in a general state. In other words, they will be reaffirmed by the results of this study. In fact, the present paper is an attempt to look forward compared with the previous ones. First, we introduce minimax modules and then we will prove some properties of them. Let $t = t_{R_+}(M,N)$ be the first integer i such that $H^i_{R_+}(M,N)$ is not minimax. We will prove that t is an element of Δ_1. Since any finitely generated module is minimax, we have $t \geq f_{R_+}(M,N) = inf\{i \mid H^i_{R_+}(M,N)\ is\ not\ finitely\ generated\}$. So, $f_{R_+}(M,N)$ is an element of Δ_1. In fact, $t_{R_+}(M,N)$ is the largest element of Δ_1 which is well known, till now. In addition we prove that:

(i) The R- module $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M,N)$ is Artinian for all $i \leq t$.

(ii) For any $i \leq t$, the set $Ass_{R_0}(H^i_{R_+}(M,N)_n)$ is asymptotically stable for $n \longrightarrow -\infty$.

(iii) For any $i \leq t$, there is a numerical polynomial $\widetilde{P} \in \mathbb{Q}[x]$ of degree less than i, such that $length_{R_0}(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)) = \widetilde{P}(n)$ for all $n \ll 0$.

Also, let $s = s_{R_+}(M,N)$ be the largest integer i such that $H^i_{R_+}(M,N)$ is not minimax. We will prove that $s_{R_+}(M,N)$ is an element of Δ_2. Since all the Artinian modules are minimax we have $s \leq a_{R_+}(M,N) = sup\{i \mid H^i_{R_+}(M,N) \textit{ is not Artinian}\}$. So, $a_{R_+}(M,N)$ is an element of Δ_2. Moreover, we will show that $H^{d-2}_{\mathfrak{m}_0}(H^s_{R_+}(M,N))$ is Artinian if and only if $H^d_{\mathfrak{m}_0}(H^{s-1}_{R_+}(M,N))$ is Artinian. Also, we prove that there is a numerical polynomial $\widetilde{Q} \in \mathbb{Q}[x]$ of degree less than s, such that $length_{R_0}(\frac{H^s_{R_+}(M,N)_n}{m_0 H^s_{R_+}(M,N)_n}) = \widetilde{Q}(n)$ for all $n \ll 0$.

We briefly recall some basic properties of generalized local cohomology.

(i) Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of finitely generated -modules, then there is a long exact sequence:

$$0 \to H^0_{R_+}(M,N') \to H^0_{R_+}(M,N) \to H^0_{R_+}(M,N'') \to H^1_{R_+}(M,N') \to \cdots$$

(ii) If N is an R_+-torsion R-module, then for each integer $i > 0$, $H^i_{R_+}(M,N) = \mathrm{Ext}^i_R(M,N)$.

(iii) If N is an m_0-torsion R-module, then for all $i \geq 0$, $H^i_{R_+}(M,N) = H^i_{\mathfrak{m}}(M,N)$ in which $\mathfrak{m} = R_0 + \mathfrak{m}_0$ is the only maximal graded ideal of R.

(iv) Let R' be a second Noetherian homogeneous graded ring and let $f : R \longrightarrow R'$ be a flat homogeneous ring homomorphism. Then $H^i_{R_+}(M,N) \cong H^i_{R'_+}(M \otimes {}_R R', N \otimes {}_R R')$ for all $i \geq 0$.

2 The results

A graded minimax R-module, is defined as follows:

Definition 2.1. *A graded R-module X is said to be a minimax module, if there is a finitely generated graded sub-module X' of X, such that $\frac{X}{X'}$ is an Artinian module.*

By the following lemma, any graded sub module and any homogeneous homomorphic image of a minimax module is minimax, too.

Lemma 2.2. *Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence of graded modules and graded homomorphisms. The module Y is minimax if and only if both of the modules X and Z are minimax.*

Proof. See [1] lemma 2.1. □

The following lemma is needed to prove most of our resultes.

Lemma 2.3. *Let X be a graded minimax module. If X is R_+-torsion, then the R-modules* $\operatorname{Tor}_j^{R_0}(\frac{R_0}{m_0}, X)$ *and* $H^j_{\mathfrak{m}_0}(X)$ *are Artinian, for all* $j \in \mathbb{N}_0$.

Proof. By definition there is a finitely generated graded sub-module X', such that $\frac{X}{X'}$ is Artinian. The exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow \frac{X}{X'} \longrightarrow 0$ induces two long exact sequences: $\operatorname{Tor}_j^{R_0}(\frac{R_0}{m_0}, X') \longrightarrow \operatorname{Tor}_j^{R_0}(\frac{R_0}{m_0}, X) \longrightarrow \operatorname{Tor}_j^{R_0}(\frac{R_0}{m_0}, \frac{X}{X'}) \longrightarrow \operatorname{Tor}_{j-1}^{R_0}(\frac{R_0}{m_0}, X')$ and $H^j_{\mathfrak{m}_0}(X') \longrightarrow H^j_{\mathfrak{m}_0}(X) \longrightarrow H^j_{\mathfrak{m}_0}(\frac{X}{X'}) \longrightarrow H^{j+1}_{\mathfrak{m}_0}(X')$. As X' is a finitely generated R_+-torsion module, $\operatorname{Tor}_j^{R_0}(\frac{R_0}{m_0}, X')$ and $H^j_{\mathfrak{m}_0}(X')$ are Artinian. Also, by [2], $\operatorname{Tor}_j^{R_0}(\frac{R_0}{m_0}, \frac{X}{X'})$ and $H^j_{\mathfrak{m}_0}(\frac{X}{X'})$ are Artinian. Therefore, the result follows easily from the above exact sequences. □

Notation 2.4. *For any graded R-modules X and Y set*

$s = s_{R_+}(M, N) = sup\{i \geq 0 \mid H^i_{R_+}(M, N) \text{ is not minimax}\}$

and

$t = t_{R_+}(M, N) = inf\{i \geq 0 \mid H^i_{R_+}(M, N) \text{ is not minimax}\}$

The main aim of this note, is to study the graded modules $H^i_{R_+}(M, N)$ and the behavior of their components $H^i_{R_+}(M, N)_n$, in the case where $i \leq t_{R_+}(M, N)$ or $i \geq s_{R_+}(M, N)$. Now, we prove a lemma which will be used for the prove of the next proposition.

Lemma 2.5. *By the above notation, the following statements hold.*

(i) $t_{R_+}(M, N) = t_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)})$ *and* $s_{R_+}(M, N) = s_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)})$.

(ii) The module $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M, N)$ *is Artinian if and only if* $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)})$ *is Artinian, For any* $i \in \mathbb{N}_0$.

Proof. Application of the functor $H^i_{R_+}(M, -)$ to the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{\mathfrak{m}_0}(N)} \longrightarrow 0$$

induces an exact sequence

$$H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0}(N)) \longrightarrow H^i_{R_+}(M, N) \xrightarrow{\eta} H^i_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)}) \longrightarrow H^{i+1}_{R_+}(M, \Gamma_{\mathfrak{m}_0}(N))$$

As $\mathfrak{m}_0 + R_+ = \mathfrak{m}$ is the only graded maximal ideal of R, the module $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0}(N)) \cong H^i_{\mathfrak{m}}(M, \Gamma_{\mathfrak{m}_0}(N))$ is Artinian for all $i \in \mathbb{N}_0$ [3,4.2]. Thus it follows from the above exact sequence that $t_{R_+}(M, N) = t_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)})$ and $s_{R_+}(M, N) = s_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)})$ and that $\ker\eta$ and $\operatorname{coker}\eta$ are Artinian modules. Now, consider the exact sequences

$$0 \longrightarrow \ker\eta \longrightarrow H^i_{R_+}(M, N) \longrightarrow \operatorname{im}\eta \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{im}\eta \longrightarrow H^i_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)}) \longrightarrow \mathrm{coker}\eta \longrightarrow 0$$

to get the following exact sequences.

$$\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \ker\eta \longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M,N) \longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \mathrm{im}\eta \longrightarrow 0$$

and

$$\mathrm{Tor}_1^{R_0}(\frac{R_0}{m_0}, \mathrm{coker}\eta) \longrightarrow \frac{R_0}{\mathfrak{m}_0}\otimes_{R_0}\mathrm{im}\eta \longrightarrow \frac{R_0}{\mathfrak{m}_0}\otimes_{R_0} H^i_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)}) \longrightarrow \frac{R_0}{\mathfrak{m}_0}\otimes_{R_0}\mathrm{coker}\eta$$

By lemma 2.3 all the ended modules of these sequences are Artinian. So, $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M,N)$ is Artinian if and only if $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \mathrm{im}\eta$ is Artinian and this is, if and only if $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M, \frac{N}{\Gamma_{\mathfrak{m}_0}(N)})$ is Artinian. □

The next proposition shows that $t_{R_+}(M,N)$ is an element of Δ_1.

Proposition 2.6. *Let $t = t_{R_+}(M,N)$. Then, for all $i \leq t$*

(i) the R-module $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M,N)$ is Artinian and

(ii) $H^j_{\mathfrak{m}_0}(H^i_{R_+}(M,N))$ is Artinian for all $0 \leq j \leq 1$.

Proof. (i) When $i < t$, the result is clear by 2.3. So, it is enough to show that $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^t_{R_+}(M,N)$ is Artinian. By lemma 2.5, we can assume that N is $\Gamma_{\mathfrak{m}_0}$-torsion-free. Thus, there is an element $x \in \mathfrak{m}_0$, such that x is a non-zero divisor on N.

The exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ induces a long exact sequence

$$\cdots \longrightarrow H^{i-1}_{R_+}(M,N) \longrightarrow H^{i-1}_{R_+}(M, \frac{N}{xN}) \longrightarrow H^i_{R_+}(M,N) \xrightarrow{x} H^i_{R_+}(M,N)$$

If $i < t$, then $H^{i-1}_{R_+}(M, \frac{N}{xN})$ is minimax, by the above sequence. So, $t_{R_+}(M, \frac{N}{xN}) \geq t-1$. Also, when $i = t$, the above long exact sequence induces an exact sequence $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^{t-1}_{R_+}(M, \frac{N}{xN}) \longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^t_{R_+}(M,N) \xrightarrow{x} \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} xH^t_{R_+}(M,N)$. As $x \in \mathfrak{m}_0$, the multiplication map "x" is zero. So, the module $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^t_{R_+}(M,N)$ is a homomorphic image of $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^{t-1}_{R_+}(M, \frac{N}{xN})$. Now, the claim (i) follows by an easy induction on t.

(ii) For all $i < t$, the result is clear by lemma 2.3. So, let $i = t$. Consider the following spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0}(H^q_{R_+}(M,N)) \underset{p}{\longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N)$$

It should be noted that $E_2^{p,q} = 0$ for all $p < 0$. So, if $0 \leq j \leq 1$, then for all $i \geq 2$ the sequence $0 \longrightarrow E_{r+1}^{j,t} \longrightarrow E_r^{j,t} \longrightarrow E_r^{j+r,t-r+1}$ is exact. In view of lemma 2.3, the right hand module of this sequence is Artinian, since $(t-r+1) < t$. Let $\{E_\infty^{p,q}\}$ be the limit term of this spectral sequence and let $r_0 \geq 2$ be an integer such that $E_{r_0+1}^{j,t} = E_{r_0+2}^{j,t} = \cdots = E_\infty^{j,t}$. As a subquotient of $H_{\mathfrak{m}}^{j+t}(M,N)$ the module $E_\infty^{j,t} = E_{r_0+1}^{j,t}$ is Artinian. Thus, from the exact sequence $0 \longrightarrow E_{r_0+1}^{j,t} \longrightarrow E_{r_0}^{j,t} \longrightarrow E_{r_0}^{j+r_0,t-r_0+1}$ and Artinianess of $E_{r_0}^{j+r_0,t-r_0+1}$, it follows that $E_{r_0}^{j,t}$ is Artinian, too. Now, repeat this argument to show that $E_{r_0-1}^{j,t}, \cdots, E_3^{j,t}$ and finally $E_2^{j,t} := H_{\mathfrak{m}_0}^j(H_{R_+}^t(M,N))$ are Artinian, as required.

□

Theorem 2.7. *Let $t = t_{R_+}(M,N)$ and let $i \leq t$, then*

(i) the set $Ass_{R_0}(H_{R_+}^i(M,N)_n)$ is asymptotically stable for $n \longrightarrow -\infty$ and

(ii) there is a numerical polynomial $\widetilde{P} \in \mathbb{Q}[x]$ of degree less than i, such that

$$length_{R_0}(\Gamma_{\mathfrak{m}_0}(H_{R_+}^i(M,N)_n)) = \widetilde{P}(n)$$

for all $n \ll 0$, and

(iii) there is a numerical polynomial $\widetilde{P'} \in \mathbb{Q}[x]$ of degree less than i, such that

$$length_{R_0}(\Gamma_{\mathfrak{m}_0}(0 \underset{H_{R_+}^i(M,N)_n}{:} \mathfrak{m}_0)) = \widetilde{P'}(n)$$

for all $n \ll 0$.

Proof. (i) Let y be an indeterminate and consider the local ring $R'_0 = R_0[y]_{\mathfrak{m}_0 R_0[y]}$ with maximal ideal $\mathfrak{m}'_0 := \mathfrak{m}_0 R'_0$, the Noetherian homogenous R'_0-algebra $R' := R \otimes_{R_0} R'_0$ and finitely generated R'-modules $M' := M \otimes_{R_0} R'_0$ and $N' := N \otimes_{R_0} R'_0$. By the flat base change property of generalized local cohomology [5], there is an isomorphism $H_{R'_+}^i(M',N')_n \cong H_{R_+}^i(M,N)_n \otimes_{R_0} R'_0$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. As R'_0 is flat over R_0, one can replace $R, \mathfrak{m}_0, M$ and N by $R', \mathfrak{m}'_0, M'$ and N' respectively and therefore assume that $\frac{R_0}{\mathfrak{m}_0}$ is infinite.

By the previous proposition the module $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^i(M,N)$ is Artinian, for all $i \leq t$. Set $\Sigma := \bigcup_{i=0}^{t} (Att_R(\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^i(M,N)) \cup Ass_R(N)) - Var(R_+)$ in which $Var(R_+)$ is the set of all graded prime ideals of R which contains R_+. As Σ is a finite set and $\frac{R_0}{\mathfrak{m}_0}$ is infinite, there exists an element $x \in R_1 - \bigcup_{P \in \Sigma} P$. It is clear that x is a non-zero divisor on N. Now, consider the exact sequence $0 \longrightarrow N(-1) \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ to get an exact sequence

$$H^{i-1}_{R_+}(M, \frac{N}{xn})_n \longrightarrow H^i_{R_+}(M,N)_{n-1} \xrightarrow{x} H^i_{R_+}(M,N)_n \longrightarrow H^i_{R_+}(M, \frac{N}{xn})_n$$

of R_0-modules. By [2, 3.2] there is some $n_0 \in \mathbb{Z} \cup \{\infty\}$ such that the multiplication map $H^i_{R_+}(M,N)_{n-1} \xrightarrow{x} H^i_{R_+}(M,N)_n$ is surjective for all $i \leq t$ and all $n \leq n_0$. So, in this case the following sequence is exact.

$$0 \longrightarrow H^{i-1}_{R_+}(M, \frac{N}{xn})_n \longrightarrow H^i_{R_+}(M,N)_{n-1} \xrightarrow{x} H^i_{R_+}(M,N)_n \longrightarrow 0 \qquad (\dagger)$$

This shows that:

$$\begin{aligned} Ass_{R_0}(H^{i-1}_{R_+}(M, \tfrac{N}{xn})_n) &\subseteq Ass_{R_0}(H^i_{R_+}(M,N)_{n-1}) \\ &\subseteq Ass_{R_0}(H^{i-1}_{R_+}(M, \tfrac{N}{xn})_n) \cup Ass_{R_0}(H^i_{R_+}(M,N)_n) \end{aligned}$$

Now, the statement (i) follows immediately by induction on $i \leq t$.

(ii) By proposition 2.6 (ii), the module $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N))$ is Artinian for all $i \leq t$. So, by [6] there exists a numerical polynomial $\widetilde{P'} \in \mathbb{Q}[x]$ such that

$$length_{R_0}(\Gamma_{\mathfrak{m}_0}(0 \underset{H^i_{R_+}(M,N)_n}{:} \mathfrak{m}_0)) = \widetilde{P'}(n)$$

for all $n \ll 0$. It remains to show that $\widetilde{P'}$ is of degree less than i. To this apply the functor $\Gamma_{\mathfrak{m}_0}(-)$ to the sequence ($\dagger$) to get the following exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0}(H^{i-1}_{R_+}(M, \frac{N}{xn})_n) \longrightarrow \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_{n-1}) \xrightarrow{x} \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)$$

hence

$$\begin{aligned} &length_{R_0}(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_{n-1})) - length_{R_0}(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)) \leq \\ &length_{R_0}(\Gamma_{\mathfrak{m}_0}(H^{i-1}_{R_+}(M, \tfrac{N}{xn})_n)) \end{aligned}$$

This allows to conclude by induction on $i \leq t$.

(iii) As a sub module of $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N))$, the module $(0 \underset{H^i_{R_+}(M,N)_n}{:} \mathfrak{m}_0)$ is Artinian for all $i \leq t$. So, the numerical polynomial $\widetilde{P'} \in \mathbb{Q}[x]$ exists again by [6]. Application of the functor $Hom_{R_0}(\frac{R_0}{\mathfrak{m}_0}, -)$ to ($\dagger$) and using similar argument mentioned in the proof (ii), yields $deg(\widetilde{P'}) < i$.

□

Proposition 2.8. *Let* $s = s_{R_+}(M,N)$. *Then* $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^i_{R_+}(M,N)$ *is Artinian for all* $i \geq s$.

Proof. By Lemma 2.3, the result is clear for all $i > s$. It remains to show that $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^s_{R_+}(M,N)$ is Artinian. To do this use induction on $n = dim_R(N)$.If $n = 0$, then N is Artinian and there is nothing to prove. So let $n > 0$ and suppose that the result has been proved for any finitely generated graded module N' with $dim_R(N') = n-1$. In view of lemma 2.5, it suffices to consider the case where $\Gamma_{\mathfrak{m}_0}(N) = 0$. Hence, there is an element $x \in \mathfrak{m}_0$, such that x is a non-zero divisor on N. Now, consider the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ to get the following exact sequence

$$0 \longrightarrow \frac{H^s_{R_+}(M,N)}{xH^s_{R_+}(M,N)} \longrightarrow H^s_{R_+}(M,\frac{N}{xN}) \longrightarrow (0 \underset{H^{s+1}_{R_+}(M,N)}{:} x) \longrightarrow 0$$

Application of the functor $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} (-)$ to this sequence induces an exact sequence

$$Tor^{R_0}_1(\frac{R_0}{\mathfrak{m}_0},(0 \underset{H^{s+1}_{R_+}(M,N)}{:} x)) \longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \frac{H^s_{R_+}(M,N)}{xH^s_{R_+}(M,N)} \longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^s_{R_+}(M,\frac{N}{xN})$$

As a submodule of $H^{s+1}_{R_+}(M,N)$, the module $(0 \underset{H^{s+1}_{R_+}(M,N)}{:} x)$ is minimax. So, the left term of the above sequence is Artinian, by lemma 2.3. Also since $dim_R(\frac{N}{xN}) = n-1$ and $s_{R_+}(M,\frac{N}{xN}) \leq s$, the right term of this sequence is Artinian, by induction hypothesis. Thus the middle term $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \frac{H^s_{R_+}(M,N)}{xH^s_{R_+}(M,N)} \cong \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^s_{R_+}(M,N)$ is Artinian, too. □

Lemma 2.9. *Let $\Gamma_{R_+}(N) = 0$ and $s = s_{R_+}(M,N)$. If $\mathfrak{m}$ is not in $Att_R(\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^s_{R_+}(M,N))$, then there is an N-regular element $x \in R_1$ such that $s_{R_+}(M,\frac{N}{xN}) \leq s-1$.*

Proof. As mentioned in the proof of 2.7, we can assume that $\frac{R_0}{\mathfrak{m}_0}$ is infinite. In view of the previous proposition, the set of attached prime ideals of $\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^s_{R_+}(M,N)$ is finite. Set $\Omega := (Att_R(\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H^s_{R_+}(M,N)) \cup Ass_R(N)) - Var(R_+)$.

Ω is a finite set of graded prime ideals of R, non of which contains R_1. As $\frac{R_0}{\mathfrak{m}_0}$ is infinite, by [4, 1.5.12] there is an element $x \in R_1$ such that x is not belong to $\bigcup_{P \in \Omega} P$. Therefore, x is a non-zero divisor on N. Use the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ to get an exact sequence

$$H^i_{R_+}(M,N) \xrightarrow{x} H^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,\frac{N}{xN}) \longrightarrow H^{i+1}_{R_+}(M,N)$$

of graded R-modules. From this sequence it follows that $H^i_{R_+}(M,\frac{N}{xN})$ is minimax for all $i > s$. So, it remains to show that $H^s_{R_+}(M,\frac{N}{xN})$ is minimax. For simplicity set $H := H^s_{R_+}(M,N)$. The fact that x is not in $\bigcup_{P \in Att_R(\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H)} P$, implies that $x\frac{H}{\mathfrak{m}_0 H} \cong$

$\frac{H}{\mathfrak{m}_0 H}$. Hence $xH_n + H_{n+1} = H_{n+1}$ for all $n \in \mathbb{Z}$. Since H_{n+1} is a finitely generated R_0-module, there is an equation $xH_n = H_{n+1}$ for all $n \in \mathbb{Z}$, by Nakayama. Therefore, the multiplication map $H \xrightarrow{x} H$ is surjective and in view of the above sequence, $H^s_{R_+}(M, \frac{N}{xN})$ is embedded in the minimax module $H^s_{R_+}(M, N)$, and this complete the proof. □

Theorem 2.10. *Let $s = s_{R_+}(M, N)$. Then there is a numerical polynomial $\widetilde{Q} \in \mathbb{Q}[x]$ of degree less than s, such that $length_{R_0}(\frac{H^s_{R_+}(M,N)_n}{\mathfrak{m}_0 H^s_{R_+}(M,N)_n}) = \widetilde{Q}(n)$ for all $n \ll 0$.*

Proof. Since $\frac{H^s_{R_+}(M,N)}{\mathfrak{m}_0 H^s_{R_+}(M,N)}$ is Artinian, the numerical polynomial $\widetilde{Q} \in \mathbb{Q}[x]$ exists by [5]. It suffices to show that $\widetilde{Q}$ is of degree less than s. Use the exact sequence $0 \longrightarrow \Gamma_{R_+}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{R_+}(N)} \longrightarrow 0$ to get a long exact sequence

$$Ext^i_R(M, \Gamma_{R_+}(N)) \longrightarrow H^i_{R_+}(M, N) \longrightarrow H^i_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)}) \longrightarrow Ext^{i+1}_R(M, \Gamma_{R_+}(N))$$

As $Ext^i_R(M, \Gamma_{R_+}(N))$ is finitely generated for all i, it follows that $s_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)}) = s$ and that $H^s_{R_+}(M, N)_n \cong H^s_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)})_n$ for all $n \ll 0$. Therefore, it suffices to consider the case where $\Gamma_{R_+}(N) = 0$. Let $\frac{H^s_{R_+}(M,N)_n}{\mathfrak{m}_0 H^s_{R_+}(M,N)} = S^1 + \cdots + S^k$ be a minimal graded secondary representation with $P_j = \sqrt{0 : S^j}$ for all $1 \leq j \leq k$. Assume that $P_k = \mathfrak{m}$. So, S^k is concentrated in finitely many degrees. Hence $length_{R_0}(\frac{H^s_{R_+}(M,N)_n}{\mathfrak{m}_0 H^s_{R_+}(M,N)_n}) = length_{R_0}(S^1_n + \cdots S^{k-1}_n)$ for all $n \ll 0$. This allows to assume that $\mathfrak{m}$ is not belong to $Att_R(\frac{H^s_{R_+}(M,N)}{\mathfrak{m}_0 H^s_{R_+}(M,N)})$. On use of previous lemma,there exists an N-regular element $x \in R_1$ such that $s_{R_+}(M, \frac{N}{xN}) \leq s-1$. So, as mentioned in the proof of the previous lemma the exact sequence $0 \longrightarrow N(-1) \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ induces an exact sequence

$$H^{s-1}_{R_+}(M, \frac{N}{xN})_n \longrightarrow H^s_{R_+}(M, N)_{n-1} \xrightarrow{x} H^s_{R_+}(M, N)_n \longrightarrow 0$$

which yields the exact sequence

$$\frac{H^{s-1}_{R_+}(M, \frac{N}{xN})_n}{\mathfrak{m}_0 H^{s-1}_{R_+}(M, \frac{N}{xN})_n} \longrightarrow \frac{H^s_{R_+}(M, N)_{n-1}}{\mathfrak{m}_0 H^s_{R_+}(M, N)_{n-1}} \longrightarrow \frac{H^s_{R_+}(M, N)_n}{\mathfrak{m}_0 H^s_{R_+}(M, N)_n} \longrightarrow 0$$

for all $n \ll 0$. This allows to conclude by induction on s.

□

Theorem 2.11. *Let $s = s_{R_+}(M, N)$ and $d = dim(R_0)$. Then*

(i) $H^j_{\mathfrak{m}_0}(H^i_{R_+}(M, N))$ *is Artinian for $d-1 \leq j \leq d$ and all $i \geq s$.*

(ii) $H^{d-2}_{\mathfrak{m}_0}(H^s_{R_+}(M,N))$ *is Artinian if and only if* $H^d_{\mathfrak{m}_0}(H^{s-1}_{R_+}(M,N))$ *is Artinian.*

Proof. (i) Consider the spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0}(H^q_{R_+}(M,N)) \underset{p}{\longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N)$$

Let $\{E_\infty^{p,q}\}$ be the limit term of this spectral sequence. As a sub quotient of $H^{p+q}_{\mathfrak{m}}(M,N)$, the module $E_\infty^{p,q}$ is Artinian for all p and q. When $i > s$ the result is clear by Lemma 2.3. So, let $i = s$ and $d-1 \leq j \leq d$. Since $E_2^{p,q} = 0$ for all $p > d$, there is an equation $E_{r+1}^{j,s} = \frac{E_r^{j,s}}{im(E_r^{j-r,s+r-1} \longrightarrow E_r^{j,s})}$ for all $r \geq 2$.

As $s+r-1 > s$ the module $L_r = im(E_r^{j-r,s+r-1} \longrightarrow E_r^{j,s})$ is Artinian, in view of Lemma 2.3. Now let $r_0 \geq 2$ be such $E_{r_0+1}^{j,s} = E_{r_0+2}^{j,s} = \cdots = E_\infty^{j,s}$. Since $E_\infty^{j,s}$ is Artinian, $E_{r_0+1}^{j,s}$ and consequently $E_{r_0}^{j,s}$ are Artinian. By repeating this argument it follows finally that $E_2^{j,s} := H^j_{\mathfrak{m}_0}(H^s_{R_+}(M,N))$ is Artinian.

(ii) Use again the previous spectral sequence to get the following exact sequence

$$0 \longrightarrow K \longrightarrow E_2^{d-2,s} \overset{d_2^{d-2,s}}{\longrightarrow} E_2^{d,s-1} \longrightarrow E_3^{d,s-1} \longrightarrow 0$$

in which $K = ker(d_2^{d-2,s})$. So, to prove the assertion, it suffices to show that both of the ended modules of this sequence are Artinian.

By definition $E_3^{d-2,s} = \frac{K}{im(E_2^{d-4,s+1} \longrightarrow E_2^{d-2,s})}$. It is easy to see that $E_\infty^{d-2,s} = \frac{E_3^{d-2,s}}{L}$ for some Artinian sub-module L of $E_3^{d-2,s}$. Therefore, using these equations and the fact that $E_\infty^{d-2,s}$ and $E_2^{d-4,s+1}$ are Artinian implies that $E_3^{d-2,s}$ and consequently the module K is Artinian. Similarly, one can show that $E_\infty^{d,s-1} = \frac{E_3^{d,s-1}}{L'}$ for some Artinian sub-module L' and conclude that $E_3^{d,s-1}$ is Artinian. □

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