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Numerical solution of nonlinear integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions

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Abstract

In this paper, we use a combination of Legendre and Block-Pulse functions on the interval $[0, 1]$ to solve the nonlinear integral equation of the second kind. The nonlinear part of the integral equation is approximated by Hybrid Legendre Block-Pulse functions, and the nonlinear integral equation is reduced to a system of nonlinear equations. We give some numerical examples. To show applicability of the proposed method.

Keywords: Legendre wavelets, Block pulse functions, Fredholm integral equations, Operational matrix.

1 Introduction

In recent years, many different basic functions have used to estimate the solution of integral equations, such as orthonormal bases and wavelets. In this paper we are going to use a simple base, that is a combination of Block-Pulse functions on $[0, 1)$, and Legendre polynomials, that is called the Hybrid Legendre Block-Pulse functions. Consider the Legendre polynomials $L_m(t)$ on the interval $[-1, 1]$:

$$\begin{aligned} &L_0(t) = 1 \\ &L_1(t) = t \\ &L_{m+1}(t) = \tfrac{2m+1}{m+1} t L_m(t) - \tfrac{m}{m+1} L_{m-1}(t) \quad m = 1, 2, 3, \ldots \end{aligned} \tag{1.1}$$

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set $\{L_m(t) : m = 0, 1, \ldots\}$ in Hilbert space $L^2[-1, 1]$ is a complete orthogonal set [3]. A set of Block-Pulse functions $b_i(t), i = 1, 2, \ldots, m$ on the interval $[0, 1)$ are defined as follows:

$$b_i(t) = \begin{cases} 1 & \frac{i-1}{m} \le t < \frac{i}{m} \\ \\ 0 & otherwise. \end{cases} \tag{1.2}$$

The Block-Pulse functions on $[0, 1)$ are disjoint, also these functions have the property of orthogonality on $[0, 1)$. For $m = 0, 1, \ldots, M-1$ and $n = 1, 2, \ldots, N$ the Hybrid Legendre Block-Pulse functions are defined as:

$$b_{n,m,t}(t) = \begin{cases} L_m(2Nt - 2n + 1) & \frac{n-1}{N} \le t < \frac{n}{N} \\ \\ 0 & otherwise. \end{cases} \tag{1.3}$$

2 Function approximation

A function $x(t) \in L^2[0, 1)$ may be expanded as:

$$x(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} X(n, m) b(n, m, t), \tag{2.1}$$

where

$$X(n, m) = \frac{(x(t), b(n, m, t))}{(b(n, m, t), b(n, m, t))}. \tag{2.2}$$

In (2.2), (. , .) denotes the inner product.
If the infinite series in (2.1) is truncated, then (2.1) can be written as:

$$x(t) \simeq x_{N,M}(t) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} X(n, m) b(n, m, t) = X^T B(t), \tag{2.3}$$

where $B(t)$ and X are $NM \times 1$ matrices given by:

$$X = [X(1, 0), X(1, 1), \ldots, X(1, M-1), \ldots, X(N, 0), \ldots, X(N, M-1)]^T. \tag{2.4}$$

and

$$B(t) = [b(1, 0, t), b(1, 1, t), \ldots, b(1, M-1, t), \ldots, b(N, 0, t), \ldots, b(N, M-1, t)]^T. \tag{2.5}$$

Similarly a function $k(t, s) \in L^2([0, 1] \times [0, 1])$ may be approximated as:

$$k(t, s) \simeq k_{N,M}(t, s) = B^T(t) K B(s); \tag{2.6}$$

where K is an $MN \times MN$ matrix such that:

$$K_{ij} = \frac{(B_i(t), ((k(t,s), B_j(s)))}{(B_i(t), B_i(t))(B_j(s), B_j(s))} \qquad for\ i,j = 1,2,\ldots,MN. \tag{2.7}$$

Also the integer powers of a function may be approximated as

$$[y(s)]^p = [Y^T B(s)]^p = B(s)^T Y_p^*. \tag{2.8}$$

Where Y_p^* is a column vector, whose elements are nonlinear combinations of the elements of the vector Y. Y_p^* is called the operational vector of the pth power of the function $y(s)$.

3 The operational matrices

The integration of the vector $B(t)$ defined in (2.5) can be obtained as:

$$\int_0^t B(s)ds = PB(t), \tag{3.1}$$

where P is an $MN \times MN$ matrix, that is called the operation matrix for Hybrid Legendre Block-Pulse functions. Then the operation matrix P has the following form [2]:

$$P = \begin{bmatrix} L & H & H & \ldots & H & H \\ 0 & L & H & \ldots & H & H \\ 0 & 0 & L & \ldots & H & H \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & L & H \\ 0 & 0 & 0 & \ldots & L & H \end{bmatrix} \tag{3.2}$$

Where H and L are $M \times M$ matrices given by:

$$H = \frac{1}{N}\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \tag{3.3}$$

also the L is a $M \times M$ matrix on the interval $[0, \frac{1}{N})$ are defined as follows [1]:

$$L = \frac{1}{2N}\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M-1} & 0 \end{bmatrix} \tag{3.4}$$

We also define the matrix D as follows:

$$D = \int_0^1 B(t)B(t)^T dt. \tag{3.5}$$

For the Hybrid Legendre Block-Pulse functions, D has the following form:

$$D = \begin{bmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_N \end{bmatrix} \tag{3.6}$$

where D_i is defined as follows:

$$D_i = \frac{1}{N}\int_0^1 L(t)L(t)^T dt. \tag{3.7}$$

4 Quadrature formulae

General Idea

We often want to calculate the inner products of functions and hybrid Legendre and Block-Pulse functions when we use Galerkin methods for nonlinear integral equation. Sweldens et al. [7] obtained a quadrature formulae for wavelet. We give a method of construction of quadrature formulae for the calculation of inner products of smooth functions and hybrid Legendre and Block-Pulse functions. The idea of quadrature formulae is to find weights $\omega_k^{(n,m)}$ and abscissae $t_k^{(n,m)}$ such that:

$$\begin{aligned} \textstyle\int_0^1 f(t)b(n,m,t)dt &= \textstyle\int_{\frac{n-1}{N}}^{\frac{n}{N}} f(t)L_m(2Nt-2n+1)dt \\ &= \textstyle\frac{1}{2N}\int_{-1}^1 f(\frac{t+2n-1}{2N})L_m(t)dt \\ &\simeq Q_r^{n,m}[f(t)] := \sum_{k=0}^{r-1}\omega_k^{(n,m)} f(t_k^{(n,m)}). \end{aligned} \tag{4.1}$$

Set

$$\mathcal{M}_p^{(n,m)} = \frac{1}{2N}\int_{-1}^1 t^p L_m(t)dt \quad\quad p \geq 0. \tag{4.2}$$

Then, we have

$$\int_0^1 t^p b(n,m,t)dt = \int_{\frac{n-1}{N}}^{\frac{(n)}{N}} t^p L_m(2Nt-2n+1))dt$$
$$= \frac{1}{2N}\int_{-1}^{1}(\tfrac{t+2n-1}{2N})^p L_m(t)dt$$
$$\simeq \frac{1}{(2N)^p}\sum_{i=0}^{p}\binom{p}{i}(2n-1)^{p-i}\mathcal{M}_i^{(n,m)}. \quad (4.3)$$

For $n = 1,2,\ldots,N,\ m = 0,1,...,M-1$ taking $\left\{t_k{}^{(n,m)}\right\}_{k=0}^{r-1}$ such that $t_k^{(n,m)} \in [\frac{n-1}{N}, \frac{n}{N}]$ by (4.1) and (4.3), we can solve the following linear equations:

$$\sum_{k=0}^{r-1}\omega_k^{(n,m)}(t_k^{(n,m)})^p = \frac{1}{(2N)^p}\sum_{i=0}^{p}\binom{p}{i}(2n-1)^{p-i}\mathcal{M}_i^{(n,m)} \qquad p=0,1,\ldots,r-1 \quad (4.4)$$

to find $\omega_k^{(n,m)}$. So, we can get MN quadrature formulae whose degree of accuracy is $r-1$. More efficient quadrature formulae can be constructed by treating the abscissae $\left\{t_k{}^{(n,m)}\right\}_{k=0}^{r-1}$ as unknowns, cf. Gauss quadrature formulae.

Calculation of $\mathcal{M}_p^{(n,m)}$

We know that the Legendre polynomials satisfy the following conditions:

$$L_m(\pm 1) = (\pm 1)^m \qquad m \geq 0$$

$$\begin{cases} L_0(t) = 1 \\ L_1(t) = t \\ L_m(t) = \frac{2m-1}{m}tL_{m-1}(t) - \frac{m-1}{m}L_{m-2}(t) \quad m \geq 2 \end{cases} \quad (4.5)$$

$$\begin{cases} L_0(t) = L_1'(t) \\ L_m(t) = \frac{L'_{m+1}(t)-L'_{m-1}(t)}{2m+1} \quad m \geq 1 \end{cases} \quad (4.6)$$

For $n = 1,2,\ldots,N,\ m = 0,1,\ldots,M-1$ and $p = 0,1,\ldots,r-1$ set

$\alpha_p^{(n,m)} = \frac{(-1)^{p+m-1}+(-1)^{p+m}}{2N(2m+1)}$.

So we have:

$$\mathcal{M}_p^{(n,m)} = \frac{1}{2N}\int_{-1}^{1} t^p L_m(t)dt$$

$$= \tfrac{1}{2N} \int_{-1}^{1} t^p \left(\tfrac{L'_{m+1}(t) - L_{m-1}(t)}{2m+1} \right) dt$$

$$= \alpha_p^{(n,m)} + \frac{p}{2m+1} \left(\int_{-1}^{1} t^{p-1} L_{m-1}(t) dt - \int_{-1}^{1} t^{p-1} L_{m+1}(t) dt \right), \tag{4.7}$$

Therefor:

$$\mathcal{M}_p^{(n,m)} = \alpha_p^{(n,m)} + \frac{p}{2(2m+1)} (M_{p-1}^{(n,m-1)} - \ M_{p-1}^{(n,m+1)}) \qquad m \geq 1. \tag{4.8}$$

Also:

$$\mathcal{M}_p^{(n,0)} = \frac{1}{2N} \int_{-1}^{1} t^p L_0(t) dt = \frac{1 + (-1)^p}{2N(p+1)}$$

$$\mathcal{M}_p^{(n,1)} = \frac{1}{2N} \int_{-1}^{1} t^p L_1(t) dt = \frac{1 - (-1)^p}{2N(p+1)} \tag{4.9}$$

When $m \geq 2$ we have:

$$\mathcal{M}_p^{(n,m)} = \tfrac{1}{2N} \int_{-1}^{1} t^p L_m(t) dt$$

$$= \tfrac{1}{2N} \int_{-1}^{1} t^p \left(\tfrac{2m-1}{m} t L_{m-1}(t) - \tfrac{m-1}{m} L_{m-2}(t) \right) dt$$

$$= \frac{2m-1}{m} \mathcal{M}_{p+1}^{(n,m-1)} - \frac{m-1}{m} \mathcal{M}_p^{(n,m-2)}. \tag{4.10}$$

5 Solution the nonlinear Fredholm integral equations

Consider the following integral equation

$$y(t) = f(t) + \int_0^1 k(t,s)[y(s)]^p ds \tag{5.1}$$

If we approximate $y(t)$, $f(t)$, $k(t,s)$ and $[y(s)]^p$ by (2.1) – (2.8) as follows:

$$y(t) = B(t)^T Y \quad f(t) \simeq B(t)^T F \quad k(t,s) \simeq B^T(t) K B(s) \quad [y(s)]^p = B(s)^T Y_p^*.$$

where Y^* is a column vector function of the elements of the vector Y. With substituting in (5.1) we have:

$$B^T(t)Y = B^T(t)F + \int_0^1 B^T(t)KB(s)B^T(s)Y^* ds = B^T(t)F + B^T(t)K(\int_0^1 B(s)B^T(s)ds)Y^* =$$

$$B^T(t)F + B^T(t)KDY^* = B^T(t)(F + KDY^*) \Longrightarrow \quad Y = F + KDY^*$$

which is a nonlinear system of equations. By solving this nonlinear system we can find the vector Y.

6 Numerical examples

We first let $M = 3$ and $N = 2$. The six basis functions are given by:

$$\left.\begin{array}{l} b(1,0,t) = 1 \\ b(1,1,t) = (4t-1) \\ b(1,2,t) = \frac{3}{2}(4t-1)^2 - \frac{1}{2} \end{array}\right\} 0 \le x < \frac{1}{2} \tag{6.1}$$

$$\left.\begin{array}{l} b(2,0,t) = 1 \\ b(2,1,t) = (4t-3) \\ b(1,2,t) = \frac{3}{2}(4t-3)^2 - \frac{1}{2} \end{array}\right\} \frac{1}{2} \le x < 1 \tag{6.2}$$

For the hybrid Legendre and Block-Pulse functions with $M = 3$ and $N = 2$ the second and tired product operation vector of y is computed as follows:

$$Y_2^* = \begin{bmatrix} y_1^2 + \frac{1}{3}y_2^2 + \frac{1}{5}y_3^2 \\ 2y_1y_2 + \frac{4}{5}y_3y_2 \\ \frac{2}{3}y_2^2 + 2y_3y_1 + \frac{2}{7}y_3^2 \\ y_4^2 + \frac{1}{3}y_5^2 + \frac{1}{5}y_6^2 \\ 2y_4y_5 + \frac{4}{5}y_5y_6 \\ \frac{2}{3}y_5^2 + 2y_4y_6 + \frac{2}{7}y_6^2 \end{bmatrix} \tag{6.3}$$

Also

$$Y_3^* = \begin{bmatrix} y_1^3 + \frac{2}{35}y_3^3 + \frac{3}{5}y_3^2y_1 + \frac{2}{5}y_3y_2^2 + y_1y_2^2 \\ 3y_1^2y_2 + \frac{3}{5}y_2^3 + \frac{33}{35}y_3^2y_2 + \frac{12}{5}y_3y_1y_2 \\ 3y_3y_1^2 + \frac{3}{7}y_3^3 + \frac{6}{7}y_3^2y_1 + \frac{11}{7}y_3y_2^2 + 2y_1y_2^2 \\ y_4^3 + \frac{2}{35}y_6^3 + \frac{3}{5}y_6^2y_4 + \frac{2}{5}y_6y_5^2 + y_4y_5^2 \\ 3y_4^2y_5 + \frac{3}{5}y_5^3 + \frac{33}{35}y_6^2y_5 + \frac{12}{5}y_6y_4y_5 \\ 3y_6y_4^2 + \frac{3}{7}y_6^3 + \frac{6}{7}y_6^2y_4 + \frac{11}{7}y_6y_5^2 + 2y_4y_5^2 \end{bmatrix} \tag{6.4}$$

Example 1:
Consider the following integral equation:

$$\left\{ \; y(t) = \frac{7}{8}x + \frac{1}{2}\int_0^1 xty(t)^2 dt \right. \tag{6.5}$$

The exact solution for this problem is $y(x) = x$. We solve (6.5) by using our method with $N = 2$ and $M = 3$. Table 1 shows the numerical results of this example, where y and $\widetilde{y}$ in the Table 1 denote the exact solution and the numerical solution, respectively.

Example 2:
Consider the following integral equation:

$$\left\{ \; y(t) = e^x - \frac{e^3}{3} - \frac{1}{3} + \int_0^1 y(t)^3 dt \right. \tag{6.6}$$

The exact solution for this problem is $y(x) = e^x$. We solve (6.6) by using our method with $N = 2$ and $M = 3$. Table 2 shows the numerical results of this example, where y and $\widetilde{y}$ in the Table 2 denote the exact solution and the numerical solution, respectively.

Example 3:
Consider the following integral equation:

$$\left\{ \; y(t) = x^2 - \frac{1}{10} + \frac{1}{2} \int_0^1 y(t)^2 dt \right. \tag{6.7}$$

The exact solution for this problem is $y(x) = x^2$. We solve (6.7) by using our method with $N = 2$ and $M = 3$. Table 3 shows the numerical results of this example, where y and $\widetilde{y}$ in the Table 3 denote the exact solution and the numerical solution, respectively.

Table 1: Numerical results of Example 1.

x_r	Exact solution $y(x_r)$	hybrid Legendre and Block-Pulse functions method $\widetilde{y}(x_r)$ $M = 3, k = 2$
0.1	0.1	0.1593726164
0.2	0.2	0.2597308849
0.3	0.3	0.3600891533
0.4	0.4	0.4604474218
0.5	0.5	0.5017913423
0.6	0.6	0.6021496108
0.7	0.7	0.7025078793
0.8	0.8	0.8028661477
0.9	0.9	0.9032244162

Table 2: Numerical results of Example 2.

x_r	Exact solution $y(x_r)$	hybrid Legendre and Block-Pulse functions method $\widetilde{y}(x_r)$ $M = 3, k = 3$
0.1	1.105170918	1.111365923
0.2	1.221402758	1.227675027
0.3	1.349858808	1.356881875
0.4	1.491824698	1.498986467
0.5	1.648721271	1.657515725
0.6	1.822118800	1.828012857
0.7	2.013752707	2.019774469
0.8	2.225540928	2.232800561
0.9	2.459603111	2.467091133

Table 3: Numerical results of Example 3.

x_r	Exact solution $y(x_r)$	hybrid Legendre and Block-Pulse functions method $\widetilde{y}(x_r)$ $M = 3, k = 2$
0.1	0.01	0.009999999997
0.2	0.04	0.040000000000
0.3	0.09	0.089999999997
0.4	0.16	0.160000000000
0.5	0.25	0.250000000000
0.6	0.36	0.360000000000
0.7	0.49	0.490000000000
0.8	0.64	0.640000000000
0.9	0.81	0.810000000000

7 conclusion

If we solve the integral equation, using orthogonal continues or piecewise constant functions, the accuracy of the method will be worse. Where as, using Hybrid Legendre and Block-Pulse functions the accuracy of system will improve using suitable M and N. Because the Hybrid Legendre and Block-Pulse functions orthogonal piecewise continues functions and they have high flexibility.

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