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Approximating fixed points of generalized non-expansive non-self mappings in $CAT(0)$ spaces

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Abstract

Suppose K is a nonempty closed convex subset of a complete $CAT(0)$ space X with the nearest point projection P from X onto K. Let $T : K \to X$ be a nonself mapping, satisfying condition (C) with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in K$, $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$,$n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then $\{x_n\}$ is $\Delta-$convergence to some point x^* in $F(T)$. This work extends a result of Laowang and Panyanak [5] to the case of generalized nonexpansive nonself mappings.

Keywords: $CAT(0)$ spaces, fixed point, generalized nonexpansive nonself mappings.

1 Introduction

In 2010 Laowang and Panyanak [5] studied the iterative scheme define as follows: Let K is a nonempty closed convex subset of a complete $CAT(0)$ space X with the nearest point projection P from X onto K and $T : K \to X$ be a nonexpansive nonself mapping with nonempty fixed point set,and if $\{x_n\}$ is generated iteratively by

$$x_1 \in K, \quad x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n]), \tag{1.1}$$

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where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then $\{x_n\}$ is $\Delta-$convergence to a fixed point of T. In this paper we extend the result of Laowang and Panyanak to the case of generalized nonexpansive nonself mappings.

Let K be a nonempty subset of a $CAT(0)$ space X and let $T : K \to X$ be a mapping. A point $x \in K$ is called a fixed point of T if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of T. T is called nonexpansive if for each $x, y \in K$, $d(Tx, Ty) \leq d(x, y)$.

In 2008, suzuki [6] introduced condition (C): T is said to satisfy condition (C) if $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$.

Proposition 1.1. *[7, Proposition 1.1] Every nonexpansive mapping satisfies condition (C), but the inverse is not true.*

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y, for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a $CAT(0)$ space (see [1]), if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{E^2}(\bar{x}, \bar{y}). \tag{1.2}$$

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \tag{CN}$$

In fact [1, page 163], a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality.

Proposition 1.2. *[6, Proposition 2.2] Let K be a bounded closed convex subset of a complete $CAT(0)$ space X and $T : K \to X$ satisfies condition (C). then*

$$d(x, Ty) \leq 3d(Tx, x) + d(x, y)$$

holds, for all $x, y \in K$.

Lemma 1.3. *Let (X, d) be a $CAT(0)$ space.*

1. *[1, Proposition 2.4] Let K be a convex subset of X which is complete in the induced metric. Then for every $x \in X$, there exists a unique point $P(x) \in K$ such that $d(x, P(x)) = \inf\{d(x, y) : y \in K\}$. Moreover, the map $x \to P(x)$ is a nonexpansive retract from X onto K.*

2. *[3, Lemma 2.1(iv)] For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*
$$d(x, z) = td(x, y), \qquad d(y, z) = (1 - t)d(x, y)$$
one uses the notation $(1 - t)x \oplus ty$ for the unique point z.

3. *[3, Lemma 2.4] For $x, y, z \in X$ and $t \in [0, 1]$, one has*
$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

4. *[3, Lemma 2.5] For $x, y, z \in X$ and $t \in [0, 1]$, one has*
$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2.$$

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X. For $x \in X$, we set
$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$
The asymptotic radius
$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$
and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set
$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$
It is known [2, Proposition 7] that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point.

Definition 1.4. [4, Definition 3.1] A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to $\Delta-$convergence to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one writes $\Delta - \lim_n x_n = x$ and call x the $\Delta - \lim$ of $\{x_n\}$.

Lemma 1.5. *Let (X, d) be a $CAT(0)$ space.*

1. *[4, p. 3690] Every bounded sequence in X has a $\Delta-$convergent subsequence.*

2. *[8, Proposition 2.1] If K is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

3. *[3, Lemma 2.8] If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

2 Main results

The following lemma was proved by Dhompongsa and Panyanak in the case of non-expansive mappings(see [3, Lemma 2.10]).

Lemma 2.1. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T : K \to X$ be a nonself mapping, satisfying condition (C). Suppose $\{x_n\}$ is a bounded below sequence in K such that $\lim_n d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By part (1) and (2) Lemma 1.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - lim_n v_n = v \in K$. We show $v \in F(T)$. In order to prove this, by the condition (C),

$$d(x_n, Tv) \leq 3d(Tx_n, x_n) + d(x_n, v).$$

Therefore

$$\begin{aligned} \limsup_n d(x_n, Tv) \leq & \limsup_n (3d(Tx_n, x_n) + d(x_n, v)) \\ = & \limsup_n d(x_n, v). \end{aligned}$$

The uniqueness of asymptotic center, imply $v \in K$ and $T(v) = v$. By part (3) Lemma 1.5, $u = v$. Therefore $\omega_w(x_n) \subset F(T)$. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, $\{d(x_n, v)\}$ converges. By part (3) Lemma 1.5, $x = u$. This shows that $\omega_w(x_n)$ consists of exactly one point. $\square$

Theorem 2.2. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X, and $T : K \to X$ be a nonself mapping, satisfying condition (C) with $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$,$n \geq 1$. then $\lim_{n\to\infty} d(x_n, x^*)$ exists.*

Proof. By part (1) of Lemma 1.3, the nearest point projection P from X onto K is nonexpansive. Then

$$\begin{aligned} d(x_{n+1}, x^*) = & \; d(P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n]), Px^*) \\ \leq & \; d((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n], x^*) \\ \leq & \; (1-\alpha_n)d(x_n, x^*) + \alpha_n d(TP[(1-\beta_n)x_n \oplus \beta_n Tx_n], Tx^*) \\ = & \; (1-\alpha_n)d(x_n, x^*) + \alpha_n d(TP[(1-\beta_n)x_n \oplus \beta_n Tx_n], x^*) \\ \leq & \; (1-\alpha_n)d(x_n, x^*) + \alpha_n(3d(Tx^*, x^*) + d(P[(1-\beta_n)x_n \oplus \beta_n Tx_n], x^*)) \\ \leq & \; (1-\alpha_n)d(x_n, x^*) + \alpha_n[(1-\beta_n)d(x_n, x^*) + \beta_n d(Tx_n, Tx^*)] \\ \leq & \; (1-\alpha_n)d(x_n, x^*) + \alpha_n[(1-\beta_n)d(x_n, x^*) + \beta_n d(x_n, x^*)] \\ = & \; d(x_n, x^*). \end{aligned}$$

Consequently we have for all $n \in N$,

$$d(x_n, x^*) \leq d(x_1, x^*).$$

This implies that $\{d(x_n, x^*)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_{n\to\infty} d(x_n, x^*)$ exists. □

Theorem 2.3. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X, and $T : K \to X$ be a nonself mapping, satisfying condition (C) with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$,$n \geq 1$. then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Let $x^* \in F(T)$. Then by Theorem 2.2, $\lim_{n\to\infty} d(x_n, x^*)$ exists. Let

$$\lim_{n\to\infty} d(x_n, x^\star) = r.$$

If $r = 0$ then by the condition (C), we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x^*, x_n) + d(x^*, Tx_n) \\ &\leq d(x^*, x_n) + 3d(x^*, Tx^*) + d(x^*, x_n) \end{aligned}$$

Therefore $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
If $r > 0$, we let $y_n = P[(1-\beta_n)x_n \oplus \beta_n Tx_n]$. By part (4) of Lemma 1.3 and condition (C), we have

$$\begin{aligned} d(y_n, x^*)^2 &= d(P[(1-\beta_n)x_n \oplus \beta_n Tx_n], Px^*)^2 \\ &\leq d((1-\beta_n)x_n \oplus \beta_n Tx_n], x^*)^2 \\ &\leq (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\ &\leq (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 \\ &= (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, Tx^*)^2 \\ &\leq (1-\beta_n)d(x_n, x^*)^2 + \beta_n(3d(Tx^*, x^*) + d(x_n, x^*))^2 \\ &= d(x_n, x^*)^2 \end{aligned}$$

Therefore

$$d(y_n, x^*) \leq d((1-\beta_n)x_n \oplus \beta_n Tx_n], x^*) \leq d(x_n, x^*) \tag{2.1}$$

By part (4) of Lemma 1.3, we have

$$\begin{aligned} d(x_{n+1}, x^*)^2 &= d(P[(1-\alpha_n)x_n \oplus \alpha_n Ty_n], Px^*)^2 \\ &\leq d((1-\alpha_n)x_n \oplus \alpha_n Ty_n, x^*)^2 \\ &\leq (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n d(Ty_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &\leq (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n(3d(Tx^*, x^*) + d(y_n, x^*))^2 \\ &\qquad - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &= (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n d(y_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &\leq (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n d(x_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &= d(x_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2. \end{aligned}$$

Therefore

$$d(x_{n+1}, x^*)^2 \leq d(x_n, x^*)^2 - W(\alpha_n) d(x_n, Ty_n)^2,$$

where $W(\alpha) = \alpha(1-\alpha)$. Since $\alpha \in [\epsilon, 1-\epsilon]$, $W(\alpha_n) \geq \epsilon^2$. Therefore

$$\epsilon^2 \sum_{n=1}^{\infty} d(x_n, Ty_n)^2 \leq d(x_1, x^*)^2 < \infty.$$

This implies $\lim_{n\to\infty} d(x_n, Ty_n) = 0$. By the condition (C), we have

$$\begin{aligned} d(x_n, x^*) \leq & \quad d(x_n, Ty_n) + d(Ty_n, x^*) \\ \leq & \quad d(x_n, Ty_n) + 3d(Tx^*, x^*) + d(y_n, x^*) \\ = & \quad d(x_n, Ty_n) + d(y_n, x^*). \end{aligned}$$

Hence

$$r \leq \liminf_{n\to\infty} d(y_n, x^*).$$

On the other hand, from (2.1), we have

$$\limsup_{n\to\infty} d(y_n, x^*) \leq r.$$

This implies $\lim_{n\to\infty} d(y_n, x^*) = r$. Thus from (2.1) we have

$$\lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n Tx_n], x^*) = r.$$

Since T satisfies in condition (C) we have

$$\begin{aligned} d(Tx_n, x^*) \leq & \quad 3d(Tx^*, x^*) + d(x_n, x^*) \\ = & \quad d(x_n, x^*) \end{aligned}$$

Thus

$$\limsup_{n\to\infty} d(Tx_n, x^*) \leq r.$$

Now, by [5, Lemma 2.9], $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Theorem 2.4. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X, and $T : K \to X$ be a nonself mapping, satisfying condition (C) with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$,$n \geq 1$. then $\{x_n\}$ $\Delta-$ converges to some point x^* in $F(T)$.*

Proof. By Theorem 2.3, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from the proof of the Theorem 2.2 that $\{d(x_n, v)\}$ is bounded and decreasing for each $v \in F(T)$, and so it is convergent. By Lemma 2.1, $\omega_w(x_n)$ consists exactly one point and is a fixed point of T. Consequently the sequence $\{x_n\}$ $\Delta-$ converges to some point x^* in $F(T)$. □

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