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Random fixed point of Meir-Keeler contraction mappings and its application

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Abstract

In this paper we introduce a generalization of Meir-Keeler contraction for random mapping $T : \Omega \times C \to C$, where C be a nonempty subset of a Banach space X and (Ω, Σ) be a measurable space with Σ being a sigma-algebra of subsets of Ω. Also, we apply such type of random fixed point results to prove the existence and unicity of a solution for an special random integral equation.

Keywords: Random fixed point, Meir-Keeler contraction, measurable space, $L-$function.
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1 Introduction

It is well known that in 1969, Meir and Keeler [1] proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point as follows:

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point.

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After that, many authors have extended, generalized and improved Banach's fixed point theorem in several ways (see for example [2]).
In this paper we proved random type of Meir-Keeler's theorem in separable Banach space. Also, we proved a new corollary to prove the existence and uniqueness of a solution for a new random integral equation.

2 Preliminaries

The following preliminaries chosen from [3, 4].
Let (Ω, Σ) be a measurable space with Σ being a sigma-algebra of subsets of Ω and let C be a nonempty subset of a Banach space X. A mapping $\xi : \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$ for each open subset U of X. The mapping $T : \Omega \times C \to C$ is a random map if and only if for each fixed $x \in C$ the mapping $T(., x) : \Omega \to C$ is measurable, and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega, .) : C \to X$ is continuous. A measurable mapping $\xi : \Omega \to X$ is a random fixed point of the random map $T : \Omega \times C \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. We denote by $RF(T)$ the set of all random fixed points of a random map T and $T_n(\omega, x)$ the n-th iteration $T(\omega, T(\omega, T(, ..., T(\omega, x))))$ of T. The letter I denotes the random mapping $I : \Omega \times C \to C$ defined by $I(\omega, x) = x$ and $T^0 = I$. We denote by $M(\Omega, X)$ the set of all measurable functions from Ω into a Banach space X.

3 Main result

Theorem 3.1. *Let X be a separable Banach space and let $T : \Omega \times X \to X$ be a mapping such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$||\xi(\omega) - \eta(\omega)|| < \varepsilon + \delta \quad \textit{implies} \quad ||T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))|| < \varepsilon \tag{3.1}$$

for all $\eta, \xi \in M(\Omega, X)$. Then T has a unique random fixed point.

Proof. One can see easily that $||T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))|| < ||\xi(\omega) - \eta(\omega)||$, for all $\eta, \xi \in M(\Omega, X)$.
Let $\xi_0 \in M(\Omega, X)$ be arbitrary and put $\xi_{n+1} = T(., \xi_n(.))$, for each $n \in \mathbb{N}$. We can show that

$$||\xi_{n+1}(\omega) - \xi_n(\omega)|| \leq ||\xi_n(\omega) - \xi_{n-1}(\omega)||. \tag{3.2}$$

For each $\omega \in \Omega$ and $n \in \mathbb{N}$. Suppose that (3.2) does not holds. Then, there exists $n_0 \in \mathbb{N}$ such that

$$||\xi_{n_0+1}(\omega) - \xi_{n_0}(\omega)|| > ||\xi_{n_0}(\omega) - \xi_{n_0-1}(\omega)||. \tag{3.3}$$

Thus for each $\delta > 0$ we have

$$||\xi_{n_0}(\omega) - \xi_{n_0-1}(\omega)|| < ||\xi_{n_0}(\omega) - \xi_{n_0+1}(\omega)|| + \delta. \tag{3.4}$$

It means that,

$$||T(\omega, \xi_{n_0}(\omega)) - T(\omega, \xi_{n_0-1}(\omega))|| < ||\xi_{n_0}(\omega) - \xi_{n_0+1}(\omega)|| \tag{3.5}$$

and this is a contradiction. Therefore, (3.2) holds. Thus, $\{\xi_n\}$ is a nondecreasing and bounded below so is convergent to ν. One can show that $\nu = 0$. Suppose that $\nu > 0$ then there exists $\delta > 0$ such that

$$||\xi_{n+1}(\omega) - \xi_n(\omega)|| < \nu + \delta \tag{3.6}$$

Thus, (3.11) shows that

$$||\xi_{n+1}(\omega) - \xi_{n+2}(\omega)|| < \nu \tag{3.7}$$

and this is a contradiction. Hence, $\nu = 0$. Now we show that $\{\xi_n\}$ is a Cauchy sequence.
If this is not, then there is a $\varepsilon > 0$ such that for all natural number k, there are $m_k, n_k > k$ so that the relation $||\xi_{m_k}(\omega) - \xi_{n_k}(\omega)|| \geq \varepsilon$. Choose a natural number M such that $||\xi_{i+1}(\omega) - \xi_i(\omega)|| < \frac{\varepsilon}{2}$ for all $i \geq M$. Also, take $m_M \geq n_M > M$ so that the relation $||\xi_{m_M}(\omega) - \xi_{n_M}(\omega)|| \geq \varepsilon$. Then,

$$\begin{aligned} ||\xi_{n_M-1}(\omega) - \xi_{n_M+1}(\omega)|| &\leq ||\xi_{n_M-1}(\omega) - \xi_{n_M}(\omega)|| + ||\xi_{n_M}(\omega) - \xi_{n_M+1}(\omega)|| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \end{aligned} \tag{3.8}$$

Hence, $||\xi_{n_M}(\omega) - \xi_{n_M+2}(\omega)|| < \frac{\varepsilon}{2}$. Similarly, $||\xi_{n_M}(\omega) - \xi_{n_M+3}(\omega)|| < \frac{\varepsilon}{2}$. Thus,

$$||\xi_{n_M}(\omega) - \xi_{m_M}(\omega)|| < \frac{\varepsilon}{2} \tag{3.9}$$

which is a contradiction. Therefore $\{\xi_n(\omega)\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is a Banach space, there is $u(\omega) \in X$ such that $\xi_n(\omega) \to u(\omega)$. Since $||T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))|| < ||\xi(\omega) - \eta(\omega)||$, for all $\xi, \eta \in X$ with $\xi \neq \eta$, thus, for each $\varepsilon \gg 0$, there is a natural number $N > 0$ such that for all $n > N$, $||\xi_n(\omega) - u(\omega)|| < \varepsilon$. Since $||T(\omega, \xi_n(\omega)) - T(\omega, u(\omega))|| < ||\xi_n(\omega) - u(\omega)||$ thus $||T(\omega, \xi_n(\omega)) - T(\omega, u(\omega))|| < \varepsilon$, for all $n > N$. It means that $T(\omega, \xi_n(\omega)) \to T(\omega, u(\omega))$. In the other side, $T(\omega, \xi_n(\omega)) = \xi_{n+1}(\omega) \to u(\omega)$ and the limit point is unique thus, $T(\omega, u(\omega)) = u(\omega)$. Now if u, v be two distinct random fixed points for T then,

$$||u(\omega) - v(\omega)|| = ||T(\omega, v(\omega)) - T(\omega, u(\omega))|| < ||u(\omega) - v(\omega)|| \tag{3.10}$$

which is a contradiction. Therefore,T has a unique random fixed point. □

Definition 3.1. *A mapping $\varphi : [0, +\infty) \to [0, +\infty)$ is called an $L-$function, if and only if, $\varphi(0) = 0$, $\varphi(t) > 0$ for each $t > 0$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t \in [\varepsilon, \varepsilon + \delta]$, $\varphi(t) \leq \varepsilon$.*

Theorem 3.2. *Let X be a separable Banach space and let $T : \Omega \times X \to X$ be a mapping such that*

$$||T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))|| \leq \varphi(||\xi(\omega) - \eta(\omega)||) \tag{3.11}$$

for all $\eta, \xi \in M(\Omega, X)$ *where,* $\varphi : [0, +\infty) \to [0, +\infty)$ *be an* $L-$*function. Then,* T *has a unique random fixed point.*

Proof. For each $\xi, \eta \in M(\Omega, X)$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq ||\xi(\omega) - \eta(\omega)|| < \varepsilon + \delta. \tag{3.12}$$

Since φ be an $L-$function thus

$$\varphi(||\xi(\omega) - \eta(\omega)||) \leq \varepsilon \tag{3.13}$$

By using (3.11) we conclude that

$$||T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))|| < \varepsilon \tag{3.14}$$

It means that, T satisfies in Meir-Keeler contraction and so T has a unique random fixed point. □

4 Random Integral Equation

Example 4.1. *Consider now the space* $X = \{x \in C([0,1]) : ||x||_\infty < \infty\}$ *and let* (Ω, Σ, p) *be a given probability space. Let* $K : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a mapping such that*

$$|K(\omega, z_1) - K(\omega, z_2)| \leq \varphi(|z_1 - z_2|) \tag{4.1}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be an* $L-$*function. Given a measurable function* ξ_0 *we are looking for solutions of the random integral equation* $\xi(\omega) = T(\omega, \xi(\omega))$ *where*

$$T(\omega, \xi(\omega)) = \xi_0(\omega) + \int_0^t K(s, [\xi(\omega)](s))\ ds. \tag{4.2}$$

This can be considered as an extension of classical Picard operator for DEs with noise on the initial condition.
It is trivial to see that $T(., \xi)$ *is measurable and* $T : \Omega \times X \to X$*. It means that,* T *is a random operator.*

$$\begin{aligned} |T(\omega, [\xi(\omega)](s)) - T(\omega, [\eta(\omega)](s))| &\leq \int_0^t |K(s, [\xi(\omega))](s) - K(s, [\eta(\omega))](s)|\ ds \\ &\leq \int_0^t \varphi(|[\xi(\omega)](s) - [\eta(\omega)](s)|)\ ds \\ &\leq \int_0^t \varphi(||\xi(\omega) - \eta(\omega)||_\infty)\ ds \\ &\leq \varphi(||\xi(\omega) - \eta(\omega)||_\infty). \end{aligned} \tag{4.3}$$

Thus,

$$||T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))||_\infty \leq \varphi(||\xi(\omega) - \eta(\omega)||_\infty) \tag{4.4}$$

and this means that T *satisfies in Theorem 3.2. Therefore,* T *has a random fixed point as a unique solution for integral equation (4.2).*

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