Mathematics Scientific Journal
Vol. 7, No. 2, (2011), 105-130

A three-step method based on Simpson's 3/8 rule for solving system of nonlinear Volterra integral equations

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Received 24 March 2011; accepted 10 April 2011

Abstract

This paper proposes a three-step method for solving nonlinear Volterra integral equations system. The proposed method convents the system to a (3×3) nonlinear block system and then by solving this nonlinear system we find approximate solution of nonlinear Volterra integral equations system. To show the advantages of our method some numerical examples are presented.

Key words: Block by block method; System of Volterra integral equations; Simpson's 3/8 rule.
2010 AMS Mathematics Subject Classification : 65F25; 65G20; 65R20

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1 Introduction

We consider the system of second Volterra integral equations (VIE) given by

$$f(x) = g(x) + \int_0^x K(x, s, f(s))\, ds, \qquad 0 \le s \le x \le X, \tag{1.1}$$

where

$$f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T, \quad g(x) = [g_1(x), g_2(x), \ldots, g_n(x)]^T,$$

and

$$K(x, s, f(s)) = \begin{bmatrix} k_{1,1}(x, s, f_1, \ldots, f_n) & \ldots\ k_{1,n}(x, s, f_1, \ldots, f_n) \\ & \vdots \\ k_{n,1}(x, s, f_1, \ldots, f_n) & \ldots\ k_{n,n}(x, s, f_1, \ldots, f_n) \end{bmatrix}.$$

Numerical solution of Volterra integral equations system has been considered by many authors.For example see [2,?,7,8,9,10]. In recent years, application of HPM (Homotopy Perturbation Method) and ADM (Adomian Decomposition Method) in nonlinear problems has been undertaken by scientists and engineers [1,3,6]. HPM [6] was proposed by He in 1998 for the first time and recently Yusufoĝlu has proposed this method [12] for solving a system of Fredholm-Volterra type integral equations. Block by block method was suggested by Young [11] for the first time in connection with product integration techniques.

In this work, we consider block by block method by using Simpson's 3/8 rule for solving linear and nonlinear systems of Volterra integral equations. This paper is organized as follow.

In section 2, we present some background material on the use of this method.

In section 3, we prove a convergence result.

Finally, numerical results are given in section 4.

2 Starting Method

Consider a system of nonlinear Volterra integral equations of the form:

$$f(x) = g(x) + \int_0^x K(x,s,f(s))\,ds, \tag{2.2}$$

where

$$f(x) = [f_1(x), f_2(x), \dots f_m(x)]^T,$$

$$K(x,s,f(s)) = \begin{bmatrix} k_{1,1}(x,s,f_1(s),\dots,f_m(s)) & \dots & k_{1,m}(x,s,f_1(s),\dots,f_m(s)) \\ k_{2,1}(x,s,f_1(s),\dots,f_m(s)) & \dots & k_{2,m}(x,s,f_1(s),\dots,f_m(s)) \\ & \vdots & \\ k_{m,1}(x,s,f_1(s),\dots,f_m(s)) & \dots & k_{m,m}(x,s,f_1(s),\dots,f_m(s)) \end{bmatrix}.$$

We suppose that the system (2.2) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of (2.2) can be found in [4]. The idea behind the block by block methods is quite general, but is most easily understood by considering a specific. Let us assume that $m = 3$ in (2.2) and use the Simpson's 3/8 rule as a numerical integration formula. Let $F_{i,j} \simeq f_i(x_j)$ then.

$$\begin{aligned} F_{1,3} \simeq f_1(x_3) = {} & g_1(x_3) + \int_0^{x_3} k_{1,1}(x_3,s,f_1(s))\,ds + \int_0^{x_3} k_{1,2}(x_3,s,f_2(s))\,ds \\ & + \int_0^{x_3} k_{1,3}(x_3,s,f_3(s))\,ds, \end{aligned} \tag{2.3}$$

$$\begin{aligned} F_{2,3} \simeq f_2(x_3) = {} & g_2(x_3) + \int_0^{x_3} k_{2,1}(x_3,s,f_1(s))\,ds + \int_0^{x_3} k_{2,2}(x_3,s,f_2(s))\,ds \\ & + \int_0^{x_3} k_{2,3}(x_3,s,f_3(s))\,ds, \end{aligned} \tag{2.4}$$

$$\begin{aligned} F_{3,3} \simeq f_3(x_3) = {} & g_3(x_3) + \int_0^{x_3} k_{3,1}(x_3,s,f_1(s))\,ds + \int_0^{x_3} k_{3,2}(x_3,s,f_2(s))\,ds \\ & + \int_0^{x_3} k_{3,3}(x_3,s,f_3(s))\,ds, \end{aligned} \tag{2.5}$$

approximating the integrals by Simpson's 3/8 rule, we have:

$$\begin{aligned} F_{1,3} &= g_1(x_3) + \tfrac{3h}{8}\{k_{1,1}(x_3,x_0,F_{1,0}) + 3k_{1,1}(x_3,x_1,F_{1,1}) + 3k_{1,1}(x_3,x_2,F_{1,2}) \\ &\quad + k_{1,1}(x_3,x_3,F_{1,3})\} + \tfrac{3h}{8}\{k_{1,2}(x_3,x_0,F_{2,0}) + 3k_{1,2}(x_3,x_1,F_{2,1}) \\ &\quad + 3k_{1,2}(x_3,x_2,F_{2,2}) + k_{1,2}(x_3,x_3,F_{2,3})\} + \tfrac{3h}{8}\{k_{1,3}(x_3,x_0,F_{3,0}) \\ &\quad + 3k_{1,3}(x_3,x_1,F_{3,1}) + 3k_{1,3}(x_3,x_2,F_{3,2}) + k_{1,3}(x_3,x_3,F_{3,3})\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} F_{2,3} &= g_2(x_3) + \tfrac{3h}{8}\{k_{2,1}(x_3,x_0,F_{1,0}) + 3k_{2,1}(x_3,x_1,F_{1,1}) + 3k_{2,1}(x_3,x_2,F_{1,2}) \\ &\quad + k_{2,1}(x_3,x_3,F_{1,3})\} + \tfrac{3h}{8}\{k_{2,2}(x_3,x_0,F_{2,0}) + 3k_{2,2}(x_3,x_1,F_{2,1}) \\ &\quad + 3k_{2,2}(x_3,x_2,F_{2,2}) + k_{2,2}(x_3,x_3,F_{2,3})\} + \tfrac{3h}{8}\{k_{2,3}(x_3,x_0,F_{3,0}) \\ &\quad + 3k_{2,3}(x_3,x_1,F_{3,1}) + 3k_{2,3}(x_3,x_2,F_{3,2}) + k_{2,3}(x_3,x_3,F_{3,3})\}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} F_{3,3} &= g_3(x_3) + \tfrac{3h}{8}\{k_{3,1}(x_3,x_0,F_{1,0}) + 3k_{3,1}(x_3,x_1,F_{1,1}) + 3k_{3,1}(x_3,x_2,F_{1,2}) \\ &\quad + k_{3,1}(x_3,x_3,F_{1,3})\} + \tfrac{3h}{8}\{k_{3,2}(x_3,x_0,F_{2,0}) + 3k_{3,2}(x_3,x_1,F_{2,1}) \\ &\quad + 3k_{3,2}(x_3,x_2,F_{2,2}) + k_{3,2}(x_3,x_3,F_{2,3})\} + \tfrac{3h}{8}\{k_{3,3}(x_3,x_0,F_{3,0}) \\ &\quad + 3k_{3,3}(x_3,x_1,F_{3,1}) + 3k_{3,3}(x_3,x_2,F_{3,2}) + k_{3,3}(x_3,x_3,F_{3,3})\}, \end{aligned} \tag{2.8}$$

where

$$F_{1,0} = g_1(x_0), \quad F_{2,0} = g_2(x_0), \quad F_{3,0} = g_3(x_0).$$

We also, get

$$\begin{aligned} F_{1,2} \simeq f_1(x_2) &= g_1(x_2) + \int_0^{x_2} k_{1,1}(x_2,s,f_1(s))\,ds + \int_0^{x_2} k_{1,2}(x_2,s,f_2(s))\,ds \\ &\quad + \int_0^{x_2} k_{1,3}(x_2,s,f_3(s))\,ds, \end{aligned} \tag{2.9}$$

$$\begin{aligned} F_{2,2} \simeq f_2(x_2) &= g_2(x_2) + \int_0^{x_2} k_{2,1}(x_2,s,f_1(s))\,ds + \int_0^{x_2} k_{2,2}(x_2,s,f_2(s))\,ds \\ &\quad + \int_0^{x_2} k_{2,3}(x_2,s,f_3(s))\,ds, \end{aligned} \tag{2.10}$$

$$F_{3,2} \simeq f_3(x_2) = g_3(x_2) + \int_0^{x_2} k_{3,1}(x_2, s, f_1(s))\,ds + \int_0^{x_2} k_{3,2}(x_2, s, f_2(s))\,ds + \int_0^{x_2} k_{3,3}(x_2, s, f_3(s))\,ds. \tag{2.11}$$

To evaluate the integrals on the right sides, we introduce points $x_{2/3} = 2h/3$, $x_{4/3} = 4h/3$ and the corresponding values $F_{2/3}$, $F_{4/3}$ and use the Simpson's 3/8 rule with step size $2h/3$, then

$$\begin{aligned} F_{1,2} &= g_1(x_2) + \tfrac{h}{4}\{k_{1,1}(x_2, x_0, F_{1,0}) + 3k_{1,1}(x_2, x_{2/3}, F_{1,2/3}) \\ &+ 3k_{1,1}(x_2, x_{4/3}, F_{1,4/3}) + k_{1,1}(x_2, x_2, F_{1,2})\} + \tfrac{h}{4}\{k_{1,2}(x_2, x_0, F_{2,0}) \\ &+ 3k_{1,2}(x_2, x_{2/3}, F_{2,2/3}) + 3k_{1,2}(x_2, x_{4/3}, F_{2,4/3}) + k_{1,2}(x_2, x_2, F_{2,2})\} \\ &+ \tfrac{h}{4}\{k_{1,3}(x_2, x_0, F_{3,0}) + 3k_{1,3}(x_2, x_{2/3}, F_{3,2/3}) + 3k_{1,3}(x_2, x_{4/3}, F_{3,4/3}) \\ &+ k_{1,3}(x_2, x_2, F_{3,2})\}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} F_{2,2} &= g_2(x_2) + \tfrac{h}{4}\{k_{2,1}(x_2, x_0, F_{1,0}) + 3k_{2,1}(x_2, x_{2/3}, F_{1,2/3}) \\ &+ 3k_{2,1}(x_2, x_{4/3}, F_{1,4/3}) + k_{2,1}(x_2, x_2, F_{1,2})\} + \tfrac{h}{4}\{k_{2,2}(x_2, x_0, F_{2,0}) \\ &+ 3k_{2,2}(x_2, x_{2/3}, F_{2,2/3}) + 3k_{2,2}(x_2, x_{4/3}, F_{2,4/3}) + k_{2,2}(x_2, x_2, F_{2,2})\} \\ &+ \tfrac{h}{4}\{k_{2,3}(x_2, x_0, F_{3,0}) + 3k_{2,3}(x_2, x_{2/3}, F_{3,2/3}) + 3k_{2,3}(x_2, x_{4/3}, F_{3,4/3}) \\ &+ k_{2,3}(x_2, x_2, F_{3,2})\}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} F_{3,2} &= g_3(x_2) + \tfrac{h}{4}\{k_{3,1}(x_2, x_0, F_{1,0}) + 3k_{3,1}(x_2, x_{2/3}, F_{1,2/3}) \\ &+ 3k_{3,1}(x_2, x_{4/3}, F_{1,4/3}) + k_{3,1}(x_2, x_2, F_{1,2})\} + \tfrac{h}{4}\{k_{3,2}(x_2, x_0, F_{2,0}) \\ &+ 3k_{3,2}(x_2, x_{2/3}, F_{2,2/3}) + 3k_{3,2}(x_2, x_{4/3}, F_{2,4/3}) + k_{3,2}(x_2, x_2, F_{2,2})\} \\ &+ \tfrac{h}{4}\{k_{3,3}(x_2, x_0, F_{3,0}) + 3k_{3,3}(x_2, x_{2/3}, F_{3,2/3}) + 3k_{3,3}(x_2, x_{4/3}, F_{3,4/3}) \\ &+ k_{3,3}(x_2, x_2, F_{3,2})\}, \end{aligned} \tag{2.14}$$

where $F_{1,2/3}$,$F_{1,4/3}$, $F_{2,2/3}$,$F_{2,4/3}$, $F_{3,2/3}$, $F_{3,4/3}$ have unknown values, which can be estimated by Lagrange interpolation points x_0, x_1, x_2, x_3. Therefore we obtain:

$$l_0(x_{2/3}) = \frac{14}{81}, l_1(x_{2/3}) = \frac{28}{27}, l_2(x_{2/3}) = \frac{-7}{27}, l_3(x_{2/3}) = \frac{4}{81},$$

$$l_0(x_{4/3}) = \frac{-5}{81}, l_1(x_{4/3}) = \frac{20}{27}, l_2(x_{4/3}) = \frac{10}{27}, l_3(x_{4/3}) = \frac{-4}{81},$$

and so

$$\begin{cases} F_{1,2/3} = \frac{14}{81}F_{1,0} + \frac{28}{27}F_{1,1} - \frac{7}{27}F_{1,2} + \frac{4}{81}F_{1,3}, \\ F_{1,4/3} = \frac{-5}{81}F_{1,0} + \frac{20}{27}F_{1,1} + \frac{10}{27}F_{1,2} - \frac{4}{81}F_{1,3}, \\ F_{2,2/3} = \frac{14}{81}F_{2,0} + \frac{28}{27}F_{2,1} - \frac{7}{27}F_{2,2} + \frac{4}{81}F_{2,3}, \\ F_{2,4/3} = \frac{-5}{81}F_{2,0} + \frac{20}{27}F_{2,1} + \frac{10}{27}F_{2,2} - \frac{4}{81}F_{2,3}, \\ F_{3,2/3} = \frac{14}{81}F_{3,0} + \frac{28}{27}F_{3,1} - \frac{7}{27}F_{3,2} + \frac{4}{81}F_{3,3}, \\ F_{3,4/3} = \frac{-5}{81}F_{3,0} + \frac{20}{27}F_{3,1} + \frac{10}{27}F_{3,2} - \frac{4}{81}F_{3,3}. \end{cases} \tag{2.15}$$

Substituting from (2.15) into (2.12), (2.13)and (2.14), we obtain the following values for $F_{1,2}$, $F_{2,2}$, $F_{3,2}$.

$$\begin{aligned} F_{1,2} = {} & g_1(x_2) + \tfrac{h}{4}\{k_{1,1}(x_2, x_0, F_{1,0}) + 3k_{1,1}(x_2, x_{2/3}, \tfrac{14}{81}F_{1,0} \\ & + \tfrac{28}{27}F_{1,1} - \tfrac{7}{27}F_{1,2} + \tfrac{4}{81}F_{1,3}) + 3k_{1,1}(x_2, x_{4/3}, -\tfrac{5}{81}F_{1,0} \\ & + \tfrac{20}{27}F_{1,1} + \tfrac{10}{27}F_{1,2} - \tfrac{4}{81}F_{1,3}) + k_{1,1}(x_2, x_2, F_{1,2})\} \\ & + \tfrac{h}{4}\{k_{1,2}(x_2, x_0, F_{2,0}) + 3k_{1,2}(x_2, x_{2/3}, \tfrac{14}{81}F_{2,0} + \tfrac{28}{27}F_{2,1} \\ & - \tfrac{7}{27}F_{2,2} + \tfrac{4}{81}F_{2,3}) + 3k_{1,2}(x_2, x_{4/3}, -\tfrac{5}{81}F_{2,0} + \tfrac{20}{27}F_{2,1} \\ & + \tfrac{10}{27}F_{2,2} - \tfrac{4}{81}F_{2,3}) + k_{1,2}(x_2, x_2, F_{2,2})\} + \tfrac{h}{4}\{k_{1,3}(x_2, x_0, F_{3,0}) \\ & + 3k_{1,3}(x_2, x_{2/3}, \tfrac{14}{81}F_{3,0} + \tfrac{28}{27}F_{3,1} - \tfrac{7}{27}F_{3,2} + \tfrac{4}{81}F_{3,3}) \\ & + 3k_{1,3}(x_2, x_{4/3}, -\tfrac{5}{81}F_{3,0} + \tfrac{20}{27}F_{3,1} + \tfrac{10}{27}F_{3,2} - \tfrac{4}{81}F_{3,3}) \\ & + k_{1,3}(x_2, x_2, F_{3,2})\}, \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
F_{2,2} = {} & g_2(x_2) + \tfrac{h}{4}\{k_{2,1}(x_2, x_0, F_{1,0}) + 3k_{2,1}(x_2, x_{2/3}, \tfrac{14}{81}F_{1,0} \\
& + \tfrac{28}{27}F_{1,1} - \tfrac{7}{27}F_{1,2} + \tfrac{4}{81}F_{1,3}) + 3k_{2,1}(x_2, x_{4/3}, -\tfrac{5}{81}F_{1,0} \\
& + \tfrac{20}{27}F_{1,1} + \tfrac{10}{27}F_{1,2} - \tfrac{4}{81}F_{1,3}) + k_{2,1}(x_2, x_2, F_{1,2})\} \\
& + \tfrac{h}{4}\{k_{2,2}(x_2, x_0, F_{2,0}) + 3k_{2,2}(x_2, x_{2/3}, \tfrac{14}{81}F_{2,0} + \tfrac{28}{27}F_{2,1} \\
& - \tfrac{7}{27}F_{2,2} + \tfrac{4}{81}F_{2,3}) + 3k_{2,2}(x_2, x_{4/3}, -\tfrac{5}{81}F_{2,0} + \tfrac{20}{27}F_{2,1} \\
& + \tfrac{10}{27}F_{2,2} - \tfrac{4}{81}F_{2,3}) + k_{2,2}(x_2, x_2, F_{2,2})\} + \tfrac{h}{4}\{k_{2,3}(x_2, x_0, F_{3,0}) \\
& + 3k_{2,3}(x_2, x_{2/3}, \tfrac{14}{81}F_{3,0} + \tfrac{28}{27}F_{3,1} - \tfrac{7}{27}F_{3,2} + \tfrac{4}{81}F_{3,3}) \\
& + 3k_{2,3}(x_2, x_{4/3}, -\tfrac{5}{81}F_{3,0} + \tfrac{20}{27}F_{3,1} + \tfrac{10}{27}F_{3,2} - \tfrac{4}{81}F_{3,3}) \\
& + k_{2,3}(x_2, x_2, F_{3,2})\},
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
F_{3,2} = {} & g_3(x_2) + \tfrac{h}{4}\{k_{3,1}(x_2, x_0, F_{1,0}) + 3k_{3,1}(x_2, x_{2/3}, \tfrac{14}{81}F_{1,0} \\
& + \tfrac{28}{27}F_{1,1} - \tfrac{7}{27}F_{1,2} + \tfrac{4}{81}F_{1,3}) + 3k_{3,1}(x_2, x_{4/3}, -\tfrac{5}{81}F_{1,0} \\
& + \tfrac{20}{27}F_{1,1} + \tfrac{10}{27}F_{1,2} - \tfrac{4}{81}F_{1,3}) + k_{3,1}(x_2, x_2, F_{1,2})\} \\
& + \tfrac{h}{4}\{k_{3,2}(x_2, x_0, F_{2,0}) + 3k_{3,2}(x_2, x_{2/3}, \tfrac{14}{81}F_{2,0} + \tfrac{28}{27}F_{2,1} \\
& - \tfrac{7}{27}F_{2,2} + \tfrac{4}{81}F_{2,3}) + 3k_{3,2}(x_2, x_{4/3}, -\tfrac{5}{81}F_{2,0} + \tfrac{20}{27}F_{2,1} \\
& + \tfrac{10}{27}F_{2,2} - \tfrac{4}{81}F_{2,3}) + k_{3,2}(x_2, x_2, F_{2,2})\} + \tfrac{h}{4}\{k_{3,3}(x_2, x_0, F_{3,0}) \\
& + 3k_{3,3}(x_2, x_{2/3}, \tfrac{14}{81}F_{3,0} + \tfrac{28}{27}F_{3,1} - \tfrac{7}{27}F_{3,2} + \tfrac{4}{81}F_{3,3}) \\
& + 3k_{3,3}(x_2, x_{4/3}, -\tfrac{5}{81}F_{3,0} + \tfrac{20}{27}F_{3,1} + \tfrac{10}{27}F_{3,2} - \tfrac{4}{81}F_{3,3}) \\
& + k_{3,3}(x_2, x_2, F_{3,2})\},
\end{aligned} \tag{2.18}
$$

also we get

$$
\begin{aligned}
F_{1,1} \simeq f_1(x_1) = {} & g_1(x_1) + \int_0^{x_1} k_{1,1}(x_1, s, f_1(s))\, ds + \int_0^{x_1} k_{1,2}(x_1, s, f_2(s))\, ds \\
& + \int_0^{x_1} k_{1,3}(x_1, s, f_3(s))\, ds,
\end{aligned} \tag{2.19}
$$

$$F_{2,1} \simeq f_2(x_1) = g_2(x_1) + \int_0^{x_1} k_{2,1}(x_1, s, f_1(s))\, ds + \int_0^{x_1} k_{2,2}(x_1, s, f_2(s))\, ds + \int_0^{x_1} k_{2,3}(x_1, s, f_3(s))\, ds, \tag{2.20}$$

$$F_{3,1} \simeq f_3(x_1) = g_3(x_1) + \int_0^{x_1} k_{3,1}(x_1, s, f_1(s))\, ds + \int_0^{x_1} k_{3,2}(x_1, s, f_2(s))\, ds + \int_0^{x_1} k_{3,3}(x_1, s, f_3(s))\, ds, \tag{2.21}$$

to evaluate the integrals on the right sides,we introduce points $x_{1/3} = h/3$, $x_{2/3} = 2h/3$ and the corresponding values $F_{1/3}$, $F_{2/3}$ and use the Simpson's 3/8 rule with step size $h/3$. Then

$$\begin{aligned} F_{1,1} &= g_1(x_1) + \tfrac{h}{8}\{k_{1,1}(x_1, x_0, F_{1,0}) + 3k_{1,1}(x_1, x_{1/3}, F_{1,1/3}) \\ &+ 3k_{1,1}(x_1, x_{2/3}, F_{1,2/3}) + k_{1,1}(x_1, x_1, F_{1,1})\} \\ &+ \tfrac{h}{8}\{k_{1,2}(x_1, x_0, F_{2,0}) + 3k_{1,2}(x_1, x_{1/3}, F_{2,1/3}) \\ &+ 3k_{1,2}(x_1, x_{2/3}, F_{2,2/3}) + k_{1,2}(x_1, x_1, F_{2,1})\} \\ &+ \tfrac{h}{8}\{k_{1,3}(x_1, x_0, F_{3,0}) + 3k_{1,3}(x_1, x_{1/3}, F_{3,1/3}) \\ &+ 3k_{1,3}(x_1, x_{2/3}, F_{3,2/3}) + k_{1,3}(x_1, x_1, F_{3,1})\} \end{aligned} \tag{2.22}$$

$$\begin{aligned} F_{2,1} &= g_2(x_1) + \tfrac{h}{8}\{k_{2,1}(x_1, x_0, F_{1,0}) + 3k_{2,1}(x_1, x_{1/3}, F_{1,1/3}) \\ &+ 3k_{2,1}(x_1, x_{2/3}, F_{1,2/3}) + k_{2,1}(x_1, x_1, F_{1,1})\} \\ &+ \tfrac{h}{8}\{k_{2,2}(x_1, x_0, F_{2,0}) + 3k_{2,2}(x_1, x_{1/3}, F_{2,1/3}) \\ &+ 3k_{2,2}(x_1, x_{2/3}, F_{2,2/3}) + k_{2,2}(x_1, x_1, F_{2,1})\} \\ &+ \tfrac{h}{8}\{k_{2,3}(x_1, x_0, F_{3,0}) + 3k_{2,3}(x_1, x_{1/3}, F_{3,1/3}) \\ &+ 3k_{2,3}(x_1, x_{2/3}, F_{3,2/3}) + k_{2,3}(x_1, x_1, F_{3,1})\} \end{aligned} \tag{2.23}$$

$$
\begin{aligned}
F_{3,1} &= g_3(x_1) + \tfrac{h}{8}\{k_{3,1}(x_1, x_0, F_{1,0}) + 3k_{3,1}(x_1, x_{1/3}, F_{1,1/3}) \\
&+ 3k_{3,1}(x_1, x_{2/3}, F_{1,2/3}) + k_{3,1}(x_1, x_1, F_{1,1})\} \\
&+ \tfrac{h}{8}\{k_{3,2}(x_1, x_0, F_{2,0}) + 3k_{3,2}(x_1, x_{1/3}, F_{2,1/3}) \\
&+ 3k_{3,2}(x_1, x_{2/3}, F_{2,2/3}) + k_{3,2}(x_1, x_1, F_{2,1})\} \\
&+ \tfrac{h}{8}\{k_{3,3}(x_1, x_0, F_{3,0}) + 3k_{3,3}(x_1, x_{1/3}, F_{3,1/3}) \\
&+ 3k_{3,3}(x_1, x_{2/3}, F_{3,2/3}) + k_{3,3}(x_1, x_1, F_{3,1})\}
\end{aligned} \tag{2.24}
$$

where $F_{1,1/3}$, $F_{1,2/3}$, $F_{2,1/3}$, $F_{2,2/3}$, $F_{3,1/3}$, $F_{3,2/3}$ have unknown values, that can be estimated by Lagrange interpolation points x_0, x_1, x_2, x_3. Therefore we obtain:

$$
l_0(x_{1/3}) = \frac{40}{81}, l_1(x_{1/3}) = \frac{20}{27}, l_2(x_{1/3}) = -\frac{8}{27}, l_3(x_{1/3}) = \frac{5}{81},
$$

$$
l_0(x_{2/3}) = \frac{14}{81}, l_1(x_{2/3}) = \frac{28}{27}, l_2(x_{2/3}) = -\frac{7}{27}, l_3(x_{2/3}) = \frac{4}{81},
$$

and so

$$
\begin{cases}
F_{1,1/3} = \frac{40}{81}F_{1,0} + \frac{20}{27}F_{1,1} - \frac{8}{27}F_{1,2} + \frac{5}{81}F_{1,3}, \\
F_{1,2/3} = \frac{14}{81}F_{1,0} + \frac{28}{27}F_{1,1} - \frac{7}{27}F_{1,2} + \frac{4}{81}F_{1,3}, \\
F_{2,1/3} = \frac{40}{81}F_{2,0} + \frac{20}{27}F_{2,1} - \frac{8}{27}F_{2,2} + \frac{5}{81}F_{2,3}, \\
F_{2,2/3} = \frac{14}{81}F_{2,0} + \frac{28}{27}F_{2,1} - \frac{7}{27}F_{2,2} + \frac{4}{81}F_{2,3}, \\
F_{3,1/3} = \frac{40}{81}F_{3,0} + \frac{20}{27}F_{3,1} - \frac{8}{27}F_{3,2} + \frac{5}{81}F_{3,3}, \\
F_{3,2/3} = \frac{14}{81}F_{3,0} + \frac{28}{27}F_{3,1} - \frac{7}{27}F_{3,2} + \frac{4}{81}F_{3,3}.
\end{cases} \tag{2.25}
$$

Substituting from (2.25) into (2.22), (2.23) and (2.24), we obtain the following values for $F_{1,1}$, $F_{2,1}$, $F_{3,1}$:

$$
\begin{aligned}
F_{1,1} &= g_1(x_1) + \tfrac{h}{8}\{k_{1,1}(x_1, x_0, F_{1,0}) + 3k_{1,1}(x_1, x_{1/3}, \tfrac{40}{81}F_{1,0} \\
&+ \tfrac{20}{27}F_{1,1} - \tfrac{8}{27}F_{1,2} + \tfrac{5}{81}F_{1,3}) + 3k_{1,1}(x_1, x_{2/3}, \tfrac{14}{81}F_{1,0} \\
&+ \tfrac{28}{27}F_{1,1} - \tfrac{7}{27}F_{1,2} + \tfrac{4}{81}F_{1,3}) + k_{1,1}(x_1, x_1, F_{1,1})\} \\
&+ \tfrac{h}{8}\{k_{1,2}(x_1, x_0, F_{2,0}) + 3k_{1,2}(x_1, x_{1/3}, \tfrac{40}{81}F_{2,0} + \tfrac{20}{27}F_{2,1} \\
&- \tfrac{8}{27}F_{2,2} + \tfrac{5}{81}F_{2,3}) + 3k_{1,2}(x_1, x_{2/3}, \tfrac{14}{81}F_{2,0} + \tfrac{28}{27}F_{2,1} \\
&- \tfrac{7}{27}F_{2,2} + \tfrac{4}{81}F_{2,3}) + k_{1,2}(x_1, x_1, F_{2,1})\} + \tfrac{h}{8}\{k_{1,3}(x_1, x_0, F_{3,0}) \\
&+ 3k_{1,3}(x_1, x_{1/3}, \tfrac{40}{81}F_{3,0} + \tfrac{20}{27}F_{3,1} - \tfrac{8}{27}F_{3,2} + \tfrac{5}{81}F_{3,3}) \\
&+ 3k_{1,3}(x_1, x_{2/3}, \tfrac{14}{81}F_{3,0} + \tfrac{28}{27}F_{3,1} - \tfrac{7}{27}F_{3,2} + \tfrac{4}{81}F_{3,3}) \\
&+ k_{1,3}(x_1, x_1, F_{3,1})\},
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
F_{2,1} &= g_2(x_1) + \tfrac{h}{8}\{k_{2,1}(x_1, x_0, F_{1,0}) + 3k_{2,1}(x_1, x_{1/3}, \tfrac{40}{81}F_{1,0} \\
&+ \tfrac{20}{27}F_{1,1} - \tfrac{8}{27}F_{1,2} + \tfrac{5}{81}F_{1,3}) + 3k_{2,1}(x_1, x_{2/3}, \tfrac{14}{81}F_{1,0} \\
&+ \tfrac{28}{27}F_{1,1} - \tfrac{7}{27}F_{1,2} + \tfrac{4}{81}F_{1,3}) + k_{2,1}(x_1, x_1, F_{1,1})\} \\
&+ \tfrac{h}{8}\{k_{2,2}(x_1, x_0, F_{2,0}) + 3k_{2,2}(x_1, x_{1/3}, \tfrac{40}{81}F_{2,0} + \tfrac{20}{27}F_{2,1} \\
&- \tfrac{8}{27}F_{2,2} + \tfrac{5}{81}F_{2,3}) + 3k_{2,2}(x_1, x_{2/3}, \tfrac{14}{81}F_{2,0} + \tfrac{28}{27}F_{2,1} \\
&- \tfrac{7}{27}F_{2,2} + \tfrac{4}{81}F_{2,3}) + k_{2,2}(x_1, x_1, F_{2,1})\} + \tfrac{h}{8}\{k_{2,3}(x_1, x_0, F_{3,0}) \\
&+ 3k_{2,3}(x_1, x_{1/3}, \tfrac{40}{81}F_{3,0} + \tfrac{20}{27}F_{3,1} - \tfrac{8}{27}F_{3,2} + \tfrac{5}{81}F_{3,3}) \\
&+ 3k_{2,3}(x_1, x_{2/3}, \tfrac{14}{81}F_{3,0} + \tfrac{28}{27}F_{3,1} - \tfrac{7}{27}F_{3,2} + \tfrac{4}{81}F_{3,3}) \\
&+ k_{2,3}(x_1, x_1, F_{3,1})\},
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
F_{3,1} = {} & g_3(x_1) + \tfrac{h}{8}\{k_{3,1}(x_1, x_0, F_{1,0}) + 3k_{3,1}(x_1, x_{1/3}, \tfrac{40}{81}F_{1,0} \\
& + \tfrac{20}{27}F_{1,1} - \tfrac{8}{27}F_{1,2} + \tfrac{5}{81}F_{1,3}) + 3k_{3,1}(x_1, x_{2/3}, \tfrac{14}{81}F_{1,0} \\
& + \tfrac{28}{27}F_{1,1} - \tfrac{7}{27}F_{1,2} + \tfrac{4}{81}F_{1,3}) + k_{3,1}(x_1, x_1, F_{1,1})\} \\
& + \tfrac{h}{8}\{k_{3,2}(x_1, x_0, F_{2,0}) + 3k_{3,2}(x_1, x_{1/3}, \tfrac{40}{81}F_{2,0} + \tfrac{20}{27}F_{2,1} \\
& - \tfrac{8}{27}F_{2,2} + \tfrac{5}{81}F_{2,3}) + 3k_{3,2}(x_1, x_{2/3}, \tfrac{14}{81}F_{2,0} + \tfrac{28}{27}F_{2,1} \qquad (2.28) \\
& - \tfrac{7}{27}F_{2,2} + \tfrac{4}{81}F_{2,3}) + k_{3,2}(x_1, x_1, F_{2,1})\} + \tfrac{h}{8}\{k_{3,3}(x_1, x_0, F_{3,0}) \\
& + 3k_{3,3}(x_1, x_{1/3}, \tfrac{40}{81}F_{3,0} + \tfrac{20}{27}F_{3,1} - \tfrac{8}{27}F_{3,2} + \tfrac{5}{81}F_{3,3}) \\
& + 3k_{3,3}(x_1, x_{2/3}, \tfrac{14}{81}F_{3,0} + \tfrac{28}{27}F_{3,1} - \tfrac{7}{27}F_{3,2} + \tfrac{4}{81}F_{3,3}) \\
& + k_{3,3}(x_1, x_1, F_{3,1})\},
\end{aligned}
$$

The (2.6), (2.7), (2.8), (2.16), (2.17), (2.18), (2.26), (2.27) and (2.28) are nine simultaneous equations in terms of unknowns $F_{1,1}$, $F_{2,1}$, $F_{3,1}$, $F_{1,2}$, $F_{2,2}$, $F_{3,2}$, $F_{1,3}$, $F_{2,3}$ and $F_{3,3}$ for the nonlinear system of VIE.
Solutions of these equations may be found by the method of successive approximation or by a suitable software package such as Maple.
For the linear system of VIE a direct method can be used for solving system of linear algebraic equations.

3 The General Process

Consider system of VIE

$$f(x) = g(x) + \int_0^x K(x, s, f(s))\, ds, \qquad 0 \le x \le a. \qquad (3.29)$$

Let $0 = x_0 < x_1 < \ldots < x_N = a$ be a partition of $[0, a]$ with the step size h, such that $x_i = ih$ for $i = 0, 1, \ldots, N$. Then we can construct a block by block approach so that a system of p simultaneous equations is obtained and thus a block of p values of F is obtained simultaneously.

We put $p = 9$ for simplicity. Setting $x = x_{3m+1}$ in (3.29) we get

$$\begin{cases} F_{1,3m+1} \simeq f_1(x_{3m+1}) \\ \qquad = g_1(x_{3m+1}) + \int_0^{x_{3m+1}} k_{1,1}(x_{3m+1}, s, f_1(s))\,ds \\ \qquad + \int_0^{x_{3m+1}} k_{1,2}(x_{3m+1}, s, f_2(s))\,ds + \int_0^{x_{3m+1}} k_{1,3}(x_{3m+1}, s, f_3(s))\,ds, \\ F_{2,3m+1} \simeq f_2(x_{3m+1}) \\ \qquad = g_2(x_{3m+1}) + \int_0^{x_{3m+1}} k_{2,1}(x_{3m+1}, s, f_1(s))\,ds \\ \qquad + \int_0^{x_{3m+1}} k_{2,2}(x_{3m+1}, s, f_2(s))\,ds + \int_0^{x_{3m+1}} k_{2,3}(x_{3m+1}, s, f_3(s))\,ds, \\ F_{3,3m+1} \simeq f_3(x_{3m+1}) \\ \qquad = g_3(x_{3m+1}) + \int_0^{x_{3m+1}} k_{3,1}(x_{3m+1}, s, f_1(s))\,ds \\ \qquad + \int_0^{x_{3m+1}} k_{3,2}(x_{3m+1}, s, f_2(s))\,ds + \int_0^{x_{3m+1}} k_{3,3}(x_{3m+1}, s, f_3(s))\,ds, \end{cases}$$

or equivalently

$$\begin{aligned} F_{1,3m+1} &= g_1(x_{3m+1}) + \int_0^{x_{3m}} k_{1,1}(x_{3m+1}, s, f_1(s))\,ds + \int_0^{x_{3m}} k_{1,2}(x_{3m+1}, s, f_2(s))\,ds \\ &+ \int_0^{x_{3m}} k_{1,3}(x_{3m+1}, s, f_3(s))\,ds + \int_{x_{3m}}^{x_{3m+1}} k_{1,1}(x_{3m+1}, s, f_1(s))\,ds \\ &+ \int_{x_{3m}}^{x_{3m+1}} k_{1,2}(x_{3m+1}, s, f_2(s))\,ds + \int_{x_{3m}}^{x_{3m+1}} k_{1,3}(x_{3m+1}, s, f_3(s))\,ds, \\ F_{2,3m+1} &= g_2(x_{3m+1}) + \int_0^{x_{3m}} k_{2,1}(x_{3m+1}, s, f_1(s))\,ds + \int_0^{x_{3m}} k_{2,2}(x_{3m+1}, s, f_2(s))\,ds \\ &+ \int_0^{x_{3m}} k_{2,3}(x_{3m+1}, s, f_3(s))\,ds + \int_{x_{3m}}^{x_{3m+1}} k_{2,1}(x_{3m+1}, s, f_1(s))\,ds \\ &+ \int_{x_{3m}}^{x_{3m+1}} k_{2,2}(x_{3m+1}, s, f_2(s))\,ds + \int_{x_{3m}}^{x_{3m+1}} k_{2,3}(x_{3m+1}, s, f_3(s))\,ds, \\ F_{3,3m+1} &= g_3(x_{3m+1}) + \int_0^{x_{3m}} k_{3,1}(x_{3m+1}, s, f_1(s))\,ds + \int_0^{x_{3m}} k_{3,2}(x_{3m+1}, s, f_2(s))\,ds \\ &+ \int_0^{x_{3m}} k_{3,3}(x_{3m+1}, s, f_3(s))\,ds + \int_{x_{3m}}^{x_{3m+1}} k_{3,1}(x_{3m+1}, s, f_1(s))\,ds \\ &+ \int_{x_{3m}}^{x_{3m+1}} k_{3,2}(x_{3m+1}, s, f_2(s))\,ds + \int_{x_{3m}}^{x_{3m+1}} k_{3,3}(x_{3m+1}, s, f_3(s))\,ds. \end{aligned}$$

Now, integration over $[0, x_{3m}]$ can be accomplished by Simpson's 3/8 rule and the integral over $[x_{3m}, x_{3m+1}]$ is computed by using a cubic interpolation. Hence

$$\begin{aligned}
F_{1,3m+1} &= g_1(x_{3m+1}) + \tfrac{3h}{8}\{k_{1,1}(x_{3m+1}, x_0, F_{1,0}) + 3k_{1,1}(x_{3m+1}, x_1, F_{1,1}) \\
&+ 3k_{1,1}(x_{3m+1}, x_2, F_{1,2}) + 2k_{1,1}(x_{3m+1}, x_3, F_{1,3}) + \ldots \\
&+ 2k_{1,1}(x_{3m+1}, x_{3m-3}, F_{1,3m-3}) + 3k_{1,1}(x_{3m+1}, x_{3m-2}, F_{1,3m-2}) \\
&+ 3k_{1,1}(x_{3m+1}, x_{3m-1}, F_{1,3m-1}) + k_{1,1}(x_{3m+1}, x_{3m}, F_{1,3m})\} \\
&+ \tfrac{3h}{8}\{k_{1,2}(x_{3m+1}, x_0, F_{2,0}) + 3k_{1,2}(x_{3m+1}, x_1, F_{2,1}) + 3k_{1,2}(x_{3m+1}, \\
&\quad x_2, F_{2,2}) + 2k_{1,2}(x_{3m+1}, x_3, F_{2,3}) + \ldots + 2k_{1,2}(x_{3m+1}, x_{3m-3}, F_{2,3m-3}) \\
&+ 3k_{1,2}(x_{3m+1}, x_{3m-2}, F_{2,3m-2}) + 3k_{1,2}(x_{3m+1}, x_{3m-1}, F_{2,3m-1}) \\
&+ k_{1,2}(x_{3m+1}, x_{3m}, F_{2,3m})\} + \tfrac{3h}{8}\{k_{1,3}(x_{3m+1}, x_0, F_{3,0}) \\
&+ 3k_{1,3}(x_{3m+1}, x_1, F_{3,1}) + 3k_{1,3}(x_{3m+1}, x_2, F_{3,2}) + 2k_{1,3}(x_{3m+1}, x_3, F_{3,3}) \\
&+ \ldots + 2k_{1,3}(x_{3m+1}, x_{3m-3}, F_{3,3m-3}) + 3k_{1,3}(x_{3m+1}, x_{3m-2}, F_{3,3m-2}) \\
&+ 3k_{1,3}(x_{3m+1}, x_{3m-1}, F_{3,3m-1}) + k_{1,3}(x_{3m+1}, x_{3m}, F_{3,3m})\} \\
&+ \tfrac{h}{8}\{k_{1,1}(x_{3m+1}, x_{3m}, F_{1,3m}) + 3k_{1,1}(x_{3m+1}, x_{3m+1/3}, \tfrac{40}{81}F_{1,3m} \\
&+ \tfrac{20}{27}F_{1,3m+1} - \tfrac{8}{27}F_{1,3m+2} + \tfrac{5}{81}F_{1,3m+3}) + 3k_{1,1}(x_{3m+1}, \\
&\quad x_{3m+2/3}, \tfrac{14}{81}F_{1,3m} + \tfrac{28}{27}F_{1,3m+1} - \tfrac{7}{27}F_{1,3m+2} + \tfrac{4}{81}F_{1,3m+3}) \\
&+ k_{1,1}(x_{3m+1}, x_{3m+1}, F_{1,3m+1})\} + \tfrac{h}{8}\{k_{1,2}(x_{3m+1}, x_{3m}, F_{2,3m}) \\
&+ 3k_{1,2}(x_{3m+1}, x_{3m+1/3}, \tfrac{40}{81}F_{2,3m} + \tfrac{20}{27}F_{2,3m+1} - \tfrac{8}{27}F_{2,3m+2} \\
&+ \tfrac{5}{81}F_{2,3m+3}) + 3k_{1,2}(x_{3m+1}, x_{3m+2/3}, \tfrac{14}{81}F_{2,3m} + \tfrac{28}{27}F_{2,3m+1} \\
&- \tfrac{7}{27}F_{2,3m+2} + \tfrac{4}{81}F_{2,3m+3}) + k_{1,2}(x_{3m+1}, x_{3m+1}, F_{2,3m+1})\} \\
&+ \tfrac{h}{8}\{k_{1,3}(x_{3m+1}, x_{3m}, F_{3,3m}) + 3k_{1,3}(x_{3m+1}, x_{3m+1/3}, \tfrac{40}{81}F_{3,3m} \\
&+ \tfrac{20}{27}F_{3,3m+1} - \tfrac{8}{27}F_{3,3m+2} + \tfrac{5}{81}F_{3,3m+3}) + 3k_{1,3}(x_{3m+1}, \\
&\quad x_{3m+2/3}, \tfrac{14}{81}F_{3,3m} + \tfrac{28}{27}F_{3,3m+1} - \tfrac{7}{27}F_{3,3m+2} + \tfrac{4}{81}F_{3,3m+3}) \\
&+ k_{1,3}(x_{3m+1}, x_{3m+1}, F_{3,3m+1})\},
\end{aligned} \tag{3.30}$$

and a similar right hand side obtains for $F_{2,3m+1}$ and $F_{3,3m+1}$, where $F_{1,0} = g_1(x_0)$, $F_{2,0} = g_2(x_0)$, $F_{3,0} = g_3(x_0)$.
Setting $x = x_{3m+2}$ in (3.29) we get

$$\begin{cases} F_{1,3m+2} \simeq f_1(x_{3m+2}) = g_1(x_{3m+2}) + \int_0^{x_{3m+2}} k_{1,1}(x_{3m+2}, s, f_1(s))\, ds \\ \qquad + \int_0^{x_{3m+2}} k_{1,2}(x_{3m+2}, s, f_2(s))\, ds + \int_0^{x_{3m+2}} k_{1,3}(x_{3m+2}, s, f_3(s))\, ds, \\ F_{2,3m+2} \simeq f_2(x_{3m+2}) = g_2(x_{3m+2}) + \int_0^{x_{3m+2}} k_{2,1}(x_{3m+2}, s, f_1(s))\, ds \\ \qquad + \int_0^{x_{3m+2}} k_{2,2}(x_{3m+2}, s, f_2(s))\, ds + \int_0^{x_{3m+2}} k_{2,3}(x_{3m+2}, s, f_3(s))\, ds, \\ F_{3,3m+2} \simeq f_3(x_{3m+2}) = g_3(x_{3m+2}) + \int_0^{x_{3m+2}} k_{3,1}(x_{3m+2}, s, f_1(s))\, ds \\ \qquad + \int_0^{x_{3m+2}} k_{3,2}(x_{3m+2}, s, f_2(s))\, ds + \int_0^{x_{3m+2}} k_{3,3}(x_{3m+2}, s, f_3(s))\, ds, \end{cases}$$

or equivalently

$$\begin{aligned} F_{1,3m+2} &= g_1(x_{3m+2}) + \int_0^{x_{3m}} k_{1,1}(x_{3m+2}, s, f_1(s))\, ds + \int_0^{x_{3m}} k_{1,2}(x_{3m+2}, s, f_2(s))\, ds \\ &+ \int_0^{x_{3m}} k_{1,3}(x_{3m+2}, s, f_3(s))\, ds + \int_{x_{3m}}^{x_{3m+2}} k_{1,1}(x_{3m+2}, s, f_1(s)) ds \\ &+ \int_{x_{3m}}^{x_{3m+2}} k_{1,2}(x_{3m+2}, s, f_2(s))\, ds + \int_{x_{3m}}^{x_{3m+2}} k_{1,3}(x_{3m+2}, s, f_3(s))\, ds, \\ F_{2,3m+2} &= g_2(x_{3m+2}) + \int_0^{x_{3m}} k_{2,1}(x_{3m+2}, s, f_1(s))\, ds + \int_0^{x_{3m}} k_{2,2}(x_{3m+2}, s, f_2(s))\, ds \\ &+ \int_0^{x_{3m}} k_{2,3}(x_{3m+2}, s, f_3(s))\, ds + \int_{x_{3m}}^{x_{3m+2}} k_{2,1}(x_{3m+2}, s, f_1(s))\, ds \\ &+ \int_{x_{3m}}^{x_{3m+2}} k_{2,2}(x_{3m+2}, s, f_2(s))\, ds + \int_{x_{3m}}^{x_{3m+2}} k_{2,3}(x_{3m+2}, s, f_3(s))\, ds, \\ F_{3,3m+2} &= g_3(x_{3m+2}) + \int_0^{x_{3m}} k_{3,1}(x_{3m+2}, s, f_1(s))\, ds + \int_0^{x_{3m}} k_{3,2}(x_{3m+2}, s, f_2(s))\, ds \\ &+ \int_0^{x_{3m}} k_{3,3}(x_{3m+2}, s, f_3(s))\, ds + \int_{x_{3m}}^{x_{3m+2}} k_{3,1}(x_{3m+2}, s, f_1(s))\, ds \\ &+ \int_{x_{3m}}^{x_{3m+2}} k_{3,2}(x_{3m+2}, s, f_2(s))\, ds + \int_{x_{3m}}^{x_{3m+2}} k_{3,3}(x_{3m+2}, s, f_3(s))\, ds. \end{aligned}$$

Now, integration over $[0, x_{3m}]$ can be accomplished by Simpson's 3/8 rule and the integral over $[x_{3m}, x_{3m+2}]$ is computed by using a cubic interpolation. Hence

$$
\begin{aligned}
F_{1,3m+2} = {} & g_1(x_{3m+2}) + \tfrac{3h}{8}\{k_{1,1}(x_{3m+2}, x_0, F_{1,0}) + 3k_{1,1}(x_{3m+2}, x_1, F_{1,1}) \\
& + 3k_{1,1}(x_{3m+2}, x_2, F_{1,2}) + 2k_{1,1}(x_{3m+2}, x_3, F_{1,3}) + \ldots \\
& + 2k_{1,1}(x_{3m+2}, x_{3m-3}, F_{1,3m-3}) + 3k_{1,1}(x_{3m+2}, x_{3m-2}, F_{1,3m-2}) \\
& + 3k_{1,1}(x_{3m+2}, x_{3m-1}, F_{1,3m-1}) + k_{1,1}(x_{3m+2}, x_{3m}, F_{1,3m})\} \\
& + \tfrac{3h}{8}\{k_{1,2}(x_{3m+2}, x_0, F_{2,0}) + 3k_{1,2}(x_{3m+2}, x_1, F_{2,1}) \\
& + 3k_{1,2}(x_{3m+2}, x_2, F_{2,2}) + 2k_{1,2}(x_{3m+2}, x_3, F_{2,3}) + \ldots \\
& + 2k_{1,2}(x_{3m+2}, x_{3m-3}, F_{2,3m-3}) + 3k_{1,2}(x_{3m+2}, x_{3m-2}, F_{2,3m-2}) \\
& + 3k_{1,2}(x_{3m+2}, x_{3m-1}, F_{2,3m-1}) + k_{1,2}(x_{3m+2}, x_{3m}, F_{2,3m})\} \\
& + \tfrac{3h}{8}\{k_{1,3}(x_{3m+2}, x_0, F_{3,0}) + 3k_{1,3}(x_{3m+2}, x_1, F_{3,1}) \\
& + 3k_{1,3}(x_{3m+2}, x_2, F_{3,2}) + 2k_{1,3}(x_{3m+2}, x_3, F_{3,3}) + \ldots \\
& + 2k_{1,3}(x_{3m+2}, x_{3m-3}, F_{3,3m-3}) + 3k_{1,3}(x_{3m+2}, x_{3m-2}, F_{3,3m-2}) \\
& + 3k_{1,3}(x_{3m+2}, x_{3m-1}, F_{3,3m-1}) + k_{1,3}(x_{3m+2}, x_{3m}, F_{3,3m})\} \\
& + \tfrac{h}{4}\{k_{1,1}(x_{3m+2}, x_{3m}, F_{1,3m}) + 3k_{1,1}(x_{3m+2}, x_{3m+2/3}, \tfrac{14}{81}F_{1,3m} \\
& + \tfrac{28}{27}F_{1,3m+1} - \tfrac{7}{27}F_{1,3m+2} + \tfrac{4}{81}F_{1,3m+3}) + 3k_{1,1}(x_{3m+2}, \\
& \quad x_{3m+4/3}, -\tfrac{5}{81}F_{1,3m} + \tfrac{20}{27}F_{1,3m+1} + \tfrac{10}{27}F_{1,3m+2} - \tfrac{4}{81}F_{1,3m+3}) \\
& + k_{1,1}(x_{3m+2}, x_{3m+2}, F_{1,3m+2})\} + \tfrac{h}{4}\{k_{1,2}(x_{3m+2}, x_{3m}, F_{2,3m}) \\
& + 3k_{1,2}(x_{3m+2}, x_{3m+2/3}, \tfrac{14}{81}F_{2,3m} + \tfrac{28}{27}F_{2,3m+1} - \tfrac{7}{27}F_{2,3m+2} \\
& + \tfrac{4}{81}F_{2,3m+3}) + 3k_{1,2}(x_{3m+2}, x_{3m+4/3}, -\tfrac{5}{81}F_{2,3m} + \tfrac{20}{27}F_{2,3m+1} \\
& + \tfrac{10}{27}F_{2,3m+2} - \tfrac{4}{81}F_{2,3m+3}) + k_{1,2}(x_{3m+2}, x_{3m+2}, F_{2,3m+2})\} \\
& + \tfrac{h}{4}\{k_{1,3}(x_{3m+2}, x_{3m}, F_{3,3m}) + 3k_{1,3}(x_{3m+2}, x_{3m+2/3}, \tfrac{14}{81}F_{3,3m} \\
& + \tfrac{28}{27}F_{3,3m+1} - \tfrac{7}{27}F_{3,3m+2} + \tfrac{4}{81}F_{3,3m+3}) + 3k_{1,3}(x_{3m+2}, \\
& \quad x_{3m+4/3}, -\tfrac{5}{81}F_{3,3m} + \tfrac{20}{27}F_{3,3m+1} + \tfrac{10}{27}F_{3,3m+2} - \tfrac{4}{81}F_{3,3m+3}) \\
& + k_{1,3}(x_{3m+2}, x_{3m+2}, F_{3,3m+2})\},
\end{aligned}
\tag{3.31}
$$

and a similar right hand side obtains for $F_{2,3m+2}$ and $F_{3,3m+2}$, where $F_{1,0} = g_1(x_0)$, $F_{2,0} = g_2(x_0)$, $F_{3,0} = g_3(x_0)$.
In a similar manner we obtain

$$\begin{cases} F_{1,3m+3} \simeq f_1(x_{3m+3}) = g_1(x_{3m+3}) + \int_0^{x_{3m+3}} k_{1,1}(x_{3m+3}, s, f_1(s))\, ds \\ \qquad + \int_0^{x_{3m+3}} k_{1,2}(x_{3m+3}, s, f_2(s))\, ds + \int_0^{x_{3m+3}} k_{1,3}(x_{3m+3}, s, f_3(s))\, ds, \\ F_{2,3m+3} \simeq f_2(x_{3m+3}) = g_2(x_{3m+3}) + \int_0^{x_{3m+3}} k_{2,1}(x_{3m+3}, s, f_1(s))\, ds \\ \qquad + \int_0^{x_{3m+3}} k_{2,2}(x_{3m+3}, s, f_2(s))\, ds + \int_0^{x_{3m+3}} k_{2,3}(x_{3m+3}, s, f_3(s))\, ds, \\ F_{3,3m+3} \simeq f_3(x_{3m+3}) = g_3(x_{3m+3}) + \int_0^{x_{3m+3}} k_{3,1}(x_{3m+3}, s, f_1(s))\, ds \\ \qquad + \int_0^{x_{3m+3}} k_{3,2}(x_{3m+3}, s, f_2(s))\, ds + \int_0^{x_{3m+3}} k_{3,3}(x_{3m+3}, s, f_3(s))\, ds, \end{cases}$$

$$\begin{aligned} F_{1,3m+3} &= g_1(x_{3m+3}) + \tfrac{3h}{8}\{k_{1,1}(x_{3m+3}, x_0, F_{1,0}) + 3k_{1,1}(x_{3m+3}, x_1, F_{1,1}) \\ &+ 3k_{1,1}(x_{3m+3}, x_2, F_{1,2}) + 2k_{1,1}(x_{3m+3}, x_3, F_{1,3}) + \dots \\ &+ 2k_{1,1}(x_{3m+3}, x_{3m}, F_{1,3m}) + 3k_{1,1}(x_{3m+3}, x_{3m+1}, F_{1,3m+1}) \\ &+ 3k_{1,1}(x_{3m+3}, x_{3m+2}, F_{1,3m+2}) + k_{1,1}(x_{3m+3}, x_{3m+3}, F_{1,3m+3})\} \\ &+ \tfrac{3h}{8}\{k_{1,2}(x_{3m+3}, x_0, F_{2,0}) + 3k_{1,2}(x_{3m+3}, x_1, F_{2,1}) \\ &+ 3k_{1,2}(x_{3m+3}, x_2, F_{2,2}) + 2k_{1,2}(x_{3m+3}, x_3, F_{2,3}) + \dots \\ &+ 2k_{1,2}(x_{3m+3}, x_{3m}, F_{2,3m}) + 3k_{1,2}(x_{3m+3}, x_{3m+1}, F_{2,3m+1}) \\ &+ 3k_{1,2}(x_{3m+3}, x_{3m+2}, F_{2,3m+2}) + k_{1,2}(x_{3m+3}, x_{3m+3}, F_{2,3m+3})\} \\ &+ \tfrac{3h}{8}\{k_{1,3}(x_{3m+3}, x_0, F_{3,0}) + 3k_{1,3}(x_{3m+3}, x_1, F_{3,1}) \\ &+ 3k_{1,3}(x_{3m+3}, x_2, F_{3,2}) + 2k_{1,3}(x_{3m+3}, x_3, F_{3,3}) + \dots \\ &+ 2k_{1,3}(x_{3m+3}, x_{3m}, F_{3,3m}) + 3k_{1,3}(x_{3m+3}, x_{3m+1}, F_{3,3m+1}) \\ &+ 3k_{1,3}(x_{3m+3}, x_{3m+2}, F_{3,3m+2}) + k_{1,3}(x_{3m+3}, x_{3m+3}, F_{3,3m+3})\}, \end{aligned} \tag{3.32}$$

$$
\begin{aligned}
F_{2,3m+3} &= g_2(x_{3m+3}) + \tfrac{3h}{8}\{k_{2,1}(x_{3m+3}, x_0, F_{1,0}) + 3k_{2,1}(x_{3m+3}, x_1, F_{1,1}) \\
&+ 3k_{2,1}(x_{3m+3}, x_2, F_{1,2}) + 2k_{2,1}(x_{3m+3}, x_3, F_{1,3}) + \ldots \\
&+ 2k_{2,1}(x_{3m+3}, x_{3m}, F_{1,3m}) + 3k_{2,1}(x_{3m+3}, x_{3m+1}, F_{1,3m+1}) \\
&+ 3k_{2,1}(x_{3m+3}, x_{3m+2}, F_{1,3m+2}) + k_{2,1}(x_{3m+3}, x_{3m+3}, F_{1,3m+3})\} \\
&+ \tfrac{3h}{8}\{k_{2,2}(x_{3m+3}, x_0, F_{2,0}) + 3k_{2,2}(x_{3m+3}, x_1, F_{2,1}) \\
&+ 3k_{2,2}(x_{3m+3}, x_2, F_{2,2}) + 2k_{2,2}(x_{3m+3}, x_3, F_{2,3}) + \ldots \\
&+ 2k_{2,2}(x_{3m+3}, x_{3m}, F_{2,3m}) + 3k_{2,2}(x_{3m+3}, x_{3m+1}, F_{2,3m+1}) \\
&+ 3k_{2,2}(x_{3m+3}, x_{3m+2}, F_{2,3m+2}) + k_{2,2}(x_{3m+3}, x_{3m+3}, F_{2,3m+3})\} \\
&+ \tfrac{3h}{8}\{k_{2,3}(x_{3m+3}, x_0, F_{3,0}) + 3k_{2,3}(x_{3m+3}, x_1, F_{3,1}) \\
&+ 3k_{2,3}(x_{3m+3}, x_2, F_{3,2}) + 2k_{2,3}(x_{3m+3}, x_3, F_{3,3}) + \ldots \\
&+ 2k_{2,3}(x_{3m+3}, x_{3m}, F_{3,3m}) + 3k_{2,3}(x_{3m+3}, x_{3m+1}, F_{3,3m+1}) \\
&+ 3k_{2,3}(x_{3m+3}, x_{3m+2}, F_{3,3m+2}) + k_{2,3}(x_{3m+3}, x_{3m+3}, F_{3,3m+3})\},
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
F_{3,3m+3} = {} & g_3(x_{3m+3}) + \tfrac{3h}{8}\{k_{3,1}(x_{3m+3}, x_0, F_{1,0}) + 3k_{3,1}(x_{3m+3}, x_1, F_{1,1}) \\
& + 3k_{3,1}(x_{3m+3}, x_2, F_{1,2}) + 2k_{3,1}(x_{3m+3}, x_3, F_{1,3}) + \ldots \\
& + 2k_{3,1}(x_{3m+3}, x_{3m}, F_{1,3m}) + 3k_{3,1}(x_{3m+3}, x_{3m+1}, F_{1,3m+1}) \\
& + 3k_{3,1}(x_{3m+3}, x_{3m+2}, F_{1,3m+2}) + k_{3,1}(x_{3m+3}, x_{3m+3}, F_{1,3m+3})\} \\
& + \tfrac{3h}{8}\{k_{3,2}(x_{3m+3}, x_0, F_{2,0}) + 3k_{3,2}(x_{3m+3}, x_1, F_{2,1}) \\
& + 3k_{3,2}(x_{3m+3}, x_2, F_{2,2}) + 2k_{3,2}(x_{3m+3}, x_3, F_{2,3}) + \ldots \\
& + 2k_{3,2}(x_{3m+3}, x_{3m}, F_{2,3m}) + 3k_{3,2}(x_{3m+3}, x_{3m+1}, F_{2,3m+1}) \\
& + 3k_{3,2}(x_{3m+3}, x_{3m+2}, F_{2,3m+2}) + k_{3,2}(x_{3m+3}, x_{3m+3}, F_{2,3m+3})\} \\
& + \tfrac{3h}{8}\{k_{3,3}(x_{3m+3}, x_0, F_{3,0}) + 3k_{3,3}(x_{3m+3}, x_1, F_{3,1}) \\
& + 3k_{3,3}(x_{3m+3}, x_2, F_{3,2}) + 2k_{3,3}(x_{3m+3}, x_3, F_{3,3}) + \ldots \\
& + 2k_{3,3}(x_{3m+3}, x_{3m}, F_{3,3m}) + 3k_{3,3}(x_{3m+3}, x_{3m+1}, F_{3,3m+1}) \\
& + 3k_{3,3}(x_{3m+3}, x_{3m+2}, F_{3,3m+2}) + k_{3,3}(x_{3m+3}, x_{3m+3}, F_{3,3m+3})\}.
\end{aligned}
\tag{3.34}
$$

The Eqs. (3.30) - (3.34) form a system with nine unknowns for $m = 1, 2, \ldots$. In fact, we have nine simultaneous equations at each step.

4 Convergence analysis

In this section we investigate the convergence of the proposed method. The following theorem shows that the order of convergence is at least four.

Theorem 4.1. The approximate method given by the systems (3.30), (3.31), (3.32), (3.33) and (3.34) is convergent and its order of convergence is at least four.

Proof. We have

$$
\begin{aligned}
|\ \varepsilon_{1,3m+1}\ | &= |\ F_{1,3m+1} - f_1(x_{3m+1})\ | \\
&= |\ h\sum_{i=0}^{3m} w_i k_{1,1}(x_{3m+1}, x_i, F_{1,i}) + h\sum_{i=0}^{3m} w_i k_{1,2}(x_{3m+1}, x_i, F_{2,i}) \\
&+ h\sum_{i=0}^{3m} w_i k_{1,3}(x_{3m+1}, x_i, F_{3,i}) \\
&+ \tfrac{h}{8} k_{1,1}(x_{3m+1}, x_{3m}, F_{1,3m}) \\
&+ \tfrac{3h}{8} k_{1,1}(x_{3m+1}, x_{3m+1/3}, \tfrac{40}{81}F_{1,3m} + \tfrac{20}{27}F_{1,3m+1} - \tfrac{8}{28}F_{1,3m+2} + \tfrac{5}{81}F_{1,3m+3}) \\
&+ \tfrac{3h}{8} k_{1,1}(x_{3m+1}, x_{3m+2/3}, \tfrac{14}{81}F_{1,3m} + \tfrac{28}{27}F_{1,3m+1} - \tfrac{7}{27}F_{1,3m+2} + \tfrac{4}{81}F_{1,3m+3}) \\
&+ \tfrac{h}{8} k_{1,1}(x_{3m+1}, x_{3m+1}, F_{1,3m+1}) + \tfrac{h}{8} k_{1,2}(x_{3m+1}, x_{3m}, F_{2,3m}) \\
&+ \tfrac{3h}{8} k_{1,2}(x_{3m+1}, x_{3m+1/3}, \tfrac{40}{81}F_{2,3m} + \tfrac{20}{27}F_{2,3m+1} - \tfrac{8}{27}F_{2,3m+2} + \tfrac{5}{81}F_{2,3m+3}) \\
&+ \tfrac{3h}{8} k_{1,2}(x_{3m+1}, x_{3m+2/3}, \tfrac{14}{81}F_{2,3m} + \tfrac{28}{27}F_{2,3m+1} - \tfrac{7}{27}F_{2,3m+2} + \tfrac{4}{81}F_{2,3m+3}) \\
&+ \tfrac{h}{8} k_{1,2}(x_{3m+1}, x_{3m+1}, F_{2,3m+1}) + \tfrac{h}{8} k_{1,3}(x_{3m+1}, x_{3m}, F_{3,3m}) \\
&+ \tfrac{3h}{8} k_{1,3}(x_{3m+1}, x_{3m+1/3}, \tfrac{40}{81}F_{3,3m} + \tfrac{20}{27}F_{3,3m+1} - \tfrac{8}{27}F_{3,3m+2} + \tfrac{5}{81}F_{3,3m+3}) \\
&+ \tfrac{3h}{8} k_{1,3}(x_{3m+1}, x_{3m+2/3}, \tfrac{14}{81}F_{3,3m} + \tfrac{28}{27}F_{3,3m+1} - \tfrac{7}{27}F_{3,3m+2} + \tfrac{4}{81}F_{3,3m+3}) \\
&+ \tfrac{h}{8} k_{1,3}(x_{3m+1}, x_{3m+1}, F_{3,3m+1}) - \int_0^{x_{3m+1}} k_{1,1}(x_{3m+1}, s, f_1(s))\, ds \\
&- \int_0^{x_{3m+1}} k_{1,2}(x_{3m+1}, s, f_2(s))\, ds - \int_0^{x_{3m+1}} k_{1,3}(x_{3m+1}, s, f_3(s))\, ds\ |,
\end{aligned}
$$

using the Lipschitz condition it can be written as

$|\ \ \varepsilon_{1,3m+1}\ \ | \leq\ hc_1\sum_{i=0}^{3m}|\varepsilon_{1,i}|\ +\ hc_2\sum_{i=0}^{3m}|\varepsilon_{2,i}|\ +\ hc_3\sum_{i=0}^{3m}|\varepsilon_{3,i}|\ +\ hc_4|\varepsilon_{1,3m+1}|\ +$
$hc_5|\varepsilon_{2,3m+1}|$

$+hc_6|\varepsilon_{3,3m+1}| + hc_7|\varepsilon_{1,3m+2}| + hc_8|\varepsilon_{2,3m+2}| + hc_9|\varepsilon_{3,3m+2}| + hc_{10}|\varepsilon_{1,3m+3}| +$
$hc_{11}|\varepsilon_{2,3m+3}|$

$+hc_{12}|\varepsilon_{3,3m+3}| + |R_{1,3m+1}| + |R_{2,3m+1}| + |R_{3,3m+1}|,$

where $R_{i,3m+1}$ $(i = 1, 2, 3)$ are the errors of integration rule.

Without less of generality, we assume that

$$\| \varepsilon_{l,j} \|_\infty = \max_{l=1,2,3} \max_{j=3m+1,3m+2,3m+3} |\varepsilon_{l,j}| = |\varepsilon_{1,3m+1}|,$$

then let $R = \max_i [|R_{1,i}|, |R_{2,i}|, |R_{3,i}|]$, hence

$$\|\varepsilon_{l,j}\| \leq hc \sum_{i=0}^{3m} (|\varepsilon_{1,i}| + |\varepsilon_{2,i}| + |\varepsilon_{3,i}|) + 9hc'\|\varepsilon_{l,j}\| + 3R,$$

and

$$\| \varepsilon_{l,j} \| \leq \frac{hc}{1-9hc'} \sum_{i=0}^{3m} (|\varepsilon_{1,i}| + |\varepsilon_{2,i}| + |\varepsilon_{3,i}|) + \frac{3R}{1-9hc'},$$

then from Gronwall inequality, we have:

$$\|\varepsilon_{l,j}\| \leq \frac{3R}{1-9hc'} e^{\frac{c}{1-9hc'} x_n}.$$

For functions k and f with at least fourth order derivatives, we have $R = o(h^4)$

and so $\| \varepsilon_m \| = o(h^4)$ and the proof is completed.

5 Numerical results

In this section, some examples are given to certify the convergence and error bounds of the presented method. All results are computed using a program written in the Maple 12. Tables 1 and 2 show the obtained results for nonlinear examples and Table 3 contain the results for linear example.

Example 1 consider the following system

Table 1
Numerical results of Example 1

x	$h = 0.1$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	$8.0e^{-09}$	$1.034e^{-08}$	$6.848092e^{-06}$
0.2	$1.0e^{-09}$	$4.1060e^{-07}$	$3.70460e^{-06}$
0.3	$5.735e^{-07}$	$1.173876e^{-05}$	$1.075440e^{-05}$
0.4	$3.5275e^{-06}$	$1.50272e^{-05}$	$1.983088e^{-05}$
0.5	$6.9509e^{-06}$	$3.55990e^{-05}$	$1.925150e^{-05}$
x	$h = 0.05$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	0	$1.57e^{-09}$	$9.8690e^{-08}$
0.2	$3.2e^{-09}$	$5.665e^{-08}$	$5.26320e^{-07}$
0.3	$3.36e^{-08}$	$9.8721e^{-07}$	$6.6720e^{-07}$
0.4	$6.85642e^{-05}$	$1.10056e^{-05}$	$1.134015e^{-05}$
0.5	$1.276489e^{-04}$	$6.4720311e^{-03}$	$2.70692661e^{-03}$
x	$h = 0.025$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	0	$2.2e^{-10}$	$1.5669e^{-08}$
0.2	$1.0651e^{-06}$	$1.3343e^{-07}$	$1.60435e^{-07}$
0.3	$2.3523836e^{-03}$	$5.215913e^{-05}$	$2.075280446e^{-02}$
0.4	$1.856403e^{-04}$	$6.2318372e^{-03}$	$1.41551977e^{-03}$
0.5	$9.845851e^{-04}$	$9.105261e^{-04}$	$2.18106048e^{-03}$

$$\begin{cases} x - x^4 + 4\int_0^x f_1(s)f_2(s)\,ds = f_1(x), \\ x^2 - x^8 + 8\int_0^x f_1(s)f_2(s)f_3(s)\,ds = f_2(x), \\ \frac{3}{4}x^4 - \frac{1}{2}x^2 - \frac{1}{5}x^5 + \int_0^x (f_1(s) + sf_2(s) + f_3(s))\,ds = f_3(x), \end{cases}$$

with the exact solutions $f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = x^4$.
Example 2. Consider the following Volterra system of integral equations:

$$\begin{cases} \sin(x) - \sin^2(x) + 2\int_0^x f_1(s)f_2(s)\,ds = f_1(x), \\ (1+x)\cos(x) - \sin(x) + \int_0^x sf_1(s)f_3^2(s)\,ds = f_2(x), \\ 2 - \cos(x) - x\sin(x) + \int_0^x sf_2(s)f_3(s)\,ds = f_3(x), \end{cases}$$

with the exact solutions $f_1(x) = \sin(x)$, $f_2(x) = \cos(x)$ and $f_3(x) = 1$.
Example 3.Consider the following Volterra system of integral equations:

$$\begin{cases} e^x - 1 + e^{-x} + \int_0^x (f_1(s) + f_2(s) + f_3(s))\,ds = f_1(x), \\ e^{-x} + 2 - 2x - 2e^x + 2\int_0^x (e^s f_2(s) - f_3(s))\,ds = f_2(x), \\ e^{2x} - e^x - 1 - 2x + 2\int_0^x e^s(f_2(s) + f_3(s))\,ds = f_3(x), \end{cases}$$

with the exact solutions $f_1(x) = e^x$, $f_2(x) = e^{-x}$ and $f_3(x) = -e^x$. The results in Tables $1-3$ show the absolute errors $|f(x_i) - Fi|$, $i = 1, 2, \ldots, N$, where $f(x_i)$ is the exact solution evaluated at $x = x_i$ and F_i is the corresponding approximate solution.

6 Conclusion

The obtained results presented in tables show the high accuracy of the proposed method. The advantages of our method are:

- in most numerical methods we need an initial value which determined by other methods. The proposed method not also gives these initial values but provide also an effective method for solving equations system on each interval;
- most of the numerical methods for a high value of x are useless, but our method is very useful;

Table 2
Numerical results of Example 2

x	$h=0.1$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	$1.2114e^{-07}$	$3.6e^{-09}$	$1.3e^{-08}$
0.2	$2.002e^{-07}$	$1.947e^{-07}$	$1.8e^{-08}$
0.3	$1.9324e^{-06}$	$1.4421e^{-06}$	$2.62e^{-07}$
0.4	$2.6889e^{-06}$	$1.3138e^{-06}$	$3.09e^{-07}$
0.5	$3.3411e^{-06}$	$1.3757e^{-06}$	$2.83e^{-07}$
x	$h=0.05$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	$2.76e^{-09}$	$5.9e^{-09}$	0
0.2	$3.85e^{-08}$	$4.65e^{-08}$	$5.0e^{-09}$
0.3	$1.215e^{-07}$	$8.95e^{-08}$	$1.6e^{-08}$
0.4	$1.0707176e^{-03}$	$2.92418e^{-05}$	$5.25729e^{-04}$
0.5	$2.3311219e^{-03}$	$6.83936153e^{-02}$	$7.58453e^{-04}$
x	$h=0.025$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	$5.7e^{-10}$	$1.2e^{-09}$	0
0.2	$6.53353e^{-05}$	$5.710e^{-07}$	$3.2788e^{-05}$
0.3	$2.90868287e^{-02}$	$3.0313694e^{-03}$	$1.90307830e^{-02}$
0.4	$1.6904541e^{-03}$	$5.99579228e^{-02}$	$5.52672e^{-04}$
0.5	$4.042949e^{-04}$	$1.3328470e^{-03}$	$1.908554e^{-03}$

Table 3
Numerical results of Example 3

x	$h = 0.1$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	$5.0132759e^{-02}$	$1.686896580e^{-01}$	$2.1443006e^{-02}$
0.2	$8.1751718e^{-02}$	$2.054755589e^{-01}$	$7.9259563e^{-02}$
0.3	$7.9479862e^{-02}$	$7.19612133e^{-02}$	$1.47756510e^{-01}$
0.4	$1.13536379e^{-01}$	$5.40168530e^{-02}$	$2.17389116e^{-01}$
0.5	$1.56802069e^{-01}$	$1.27864643e^{-02}$	$3.11108432e^{-01}$
x	$h = 0.05$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	$2.8019196e^{-02}$	$8.82301280e^{-02}$	$1.5655539e^{-02}$
0.2	$1.7504754e^{-02}$	$1.72855609e^{-02}$	$3.0117878e^{-02}$
0.3	$2.4855094e^{-02}$	$1.40649833e^{-02}$	$4.3619582e^{-02}$
0.4	$4.2075731e^{-02}$	$2.44063830e^{-02}$	$8.3512095e^{-02}$
0.5	$3.2844096e^{-02}$	$8.63902173e^{-02}$	$5.8645790e^{-02}$
x	$h = 0.025$		
	$e(f_1)$	$e(f_2)$	$e(f_3)$
0.1	$3.410525e^{-03}$	$4.3355870e^{-03}$	$5.629077e^{-03}$
0.2	$6.647754e^{-03}$	$7.5981789e^{-03}$	$1.2226518e^{-02}$
0.3	$5.7400509e^{-02}$	$3.339441417e^{-01}$	$1.00418582e^{-01}$
0.4	$2.940787e^{-03}$	$1.107626290e^{-01}$	$4.854073e^{-03}$
0.5	$4.971338e^{-03}$	$5.52297317e^{-02}$	$7.129213e^{-03}$

- all unknowns are obtained at the same time and with increasing the p value we find more unknowns;
- at each stage we find 9 unknowns simultaneously and
- the convergence order is at least 4.

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