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Common fixed point theorem for w-distance with new integral type contraction

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Abstract

Boujari [5] proved a fixed point theorem with an old version of the integral type contraction , his proof is incorrect. In this paper, a new generalization of integral type contraction is introduced. Moreover, a fixed point theorem is obtained.

Key words: Fixed point; Common fixed point; Integral type contraction; w-distance.

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1 Introduction

The first important result in fixed point theory is Banach's contraction principle. Branciari [6] established a fixed point result for an integral-type inequality, that is generalization of Banach's contraction principle. The concept of a w-distance on a metric space was introduced by Kada et al [8] to generalize some important results in nonconvex minimizations. Literature abounds with several contractive conditions that have been employed by various researchers over the years to obtain different fixed point theorems. For various contractive definitions that have been employed, we refer our interested readers to Berinde [2,**?**], Branciari [6], Rhoades [10], Razani et al [**?**,9], Asadi et al [1], Ghoncheh et al [7] and Shabani et al [11].
Branciari [6], Bujari [5] and Rhoades [10] used contractive conditions of integral type to extend Banach's fixed point theorem. In this paper, we shall establish some common fixed point theorems by employing the concepts of an w-distance as well as one contractive condition of integral type. A corrected version of Theorem 2.1 in Bujari [5] is given.

In 1996 , Kada and et al in [8] for the first time introduced the concept of w-distance on a metric space.

Definition 1.1 Let X be a metric space with metric d. Then a function $p : X \times X \longrightarrow [0, \infty)$ is called w-distance on X if the following satisfy:
(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$
(2) for any $x \in X$, $p(x, .) \longrightarrow [0, \infty)$ is lower semi-continuous.
(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Definition 1.2 Let (X, d) be a metric space and p be a w-distance on X.
(i) X is said to be p-bounded if $sup\{p(x, y)|x, y \in X\} < \infty$.
(ii) $f : X \to X$ is p-continuous if $lim_{n\to\infty}p(x_n, x) = 0$ implies

$$lim_{n\to\infty}p(f(x_n), f(x)) = 0.$$

(iii) Sequence $\{x_n\}_{n=1}^{\infty}$ is p-Cauchy if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$,

if $m,n \geq N$ then $p(x_n, x_m) < \varepsilon$.
(iv) X is said to be p-complete if for every p-Cauchy sequence $\{x_n\}_{n=1}^{\infty}$; there exists $x \in X$ with $lim_{n\to\infty} p(x_n, x) = 0$.

Lemma 1.1 *(See [8]) Let (X,d) be a metric space and p be a w-distance on X. If $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = 0$ then $x = y$. In particular, if $p(z,x) = p(z,y) = 0$ then $x = y$.*

Also, Branciari [6] established a fixed point result for an integral-type inequality, that is generalization Banach's contraction principle. Baraciari [6] proved the following fixed point theorem.

Theorem 1.1 *Let (X,d) be a complete metric space,$c \in]0,1[$, and $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable on each compact subset of $\mathbb{R}^+$,nonnegative, and for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0.$$

then f has a unique fixed point $a \in X$ such that $lim_{n\to\infty} f^n(x) = a$, for each $x \in X$.

2 Main result

The next theorem is the generalization of Theorem 2.1 of Boujari [5].

Theorem 2.1 *Let (X,d) be a metric space and p be a w-distance on X. Suppose X be a p-complete and p-bounded. Let f and g commuting, $f(X) \subset g(X)$, p-continuous and satisfying*

$$\int_0^{p(f(x),f(y))} \phi(t)dt \leq (1-k_n) \int_0^{p(g(x),g(y))} \phi(t)dt \qquad (2.1)$$

for each $x, y \in X$, where real numbers $\{k_n\}_{n=1}^{\infty}$ in $[0,1]$ with $\sum_{n=1}^{\infty} k_n = \infty$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, nonnegative, and for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

then f and g have a unique common fixed point.

Proof. Let $x_0, x_1 \in X$ be such that $g(x_1) = f(x_0)$ and choose x_n such that, $g(x_n) = f(x_{n-1})$.
We obtain by (2.1)

$$\begin{aligned}
\int_0^{p(f(x_n),f(x_{n+m}))} \phi(t)dt &\leq (1-k_1) \int_0^{p(g(x_n),g(x_{n+m}))} \phi(t)dt \\
&\leq e^{-k_1} \int_0^{p(f(x_{n-1}),f(x_{n+m-1}))} \phi(t)dt \\
&\leq e^{-k_1}(1-k_2) \int_0^{p(g(x_{n-1}),g(x_{n+m-1}))} \phi(t)dt \\
&\leq e^{-k_1} e^{-k_2} \int_0^{p(f(x_{n-2}),f(x_{n+m-2}))} \phi(t)dt \\
&\;\;\vdots \\
&\leq e^{-\sum_{j=1}^{n} k_j} \int_0^{p(f(x_0),f(x_m))} \phi(t)dt.
\end{aligned}$$

X is p-bounded and for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. $p(f(x_0), f(x_m))$ is finite number and if $n \to \infty$, then $e^{-\sum_{j=1}^{n} k_j} \to 0$ for which it follow that

$$\int_0^{p(f(x_n),f(x_{n+m}))} \phi(t)dt \to 0 \text{ as } n \to \infty.$$

so that $lim_{n\to\infty} p(f(x_n), f(x_{n+m})) = 0$, then $\{f(x_n)\}$ is p-Cauchy and with p-complete of X for some $x \in X$ we have $lim_{n\to\infty} p(f(x_n), x) = 0$, then $lim_{n\to\infty} p(g(x_n), x) = 0$. Since f and g are p-continuous

$$lim_{n\to\infty} p(f(g(x_n)), f(x)) = lim_{n\to\infty} p(g(f(x_n)), g(x)) = 0.$$

Also, since f and g are commuting, then $fg = gf$, so that

$$lim_{n\to\infty} p(f(g(x_n)), f(x)) = lim_{n\to\infty} p(f(g(x_n)), g(x)) = 0.$$

Thus definition of w-distance implies $f(x) = g(x), fg = gf$ and so $f(f(x)) = f(g(x)) = g(f(x)) = g(g(x))$.

Suppose $p(f(x), f(f(x)) \neq 0$. Using contraction (2.1),

$$\begin{aligned} \int_0^{p(f(x),f(f(x)))} \phi(t)dt &\leq (1-k_n) \int_0^{p(g(x),g(f(x)))} \phi(t)dt \\ &= (1-k_n) \int_0^{p(f(x),f(f(x))} \phi(t)dt \end{aligned}$$

for which

$$k_n \int_0^{p(f(x),f(f(x)))} \phi(t)dt \leq 0.$$

Since $k_n \in [0,1]$, one can obtain $\int_0^{p(f(x),f(f(x)))} \phi(t)dt \leq 0$, which is a contradiction since p is nonnegative. Therefore, $\int_0^{p(f(x),f(f(x)))} \phi(t)dt = 0$. From which it follows that $p(f(x), f(f(x))) = 0$.
Similarly, suppose that $p(f(x), f(x)) \neq 0$. Using contraction (2.1) again we have

$$\int_0^{p(f(x),f(x))} \phi(t)dt \leq (1-k_n) \int_0^{p(f(x),f(x))} \phi(t)dt$$

from which we obtain $\int_0^{p(f(x),f(x))} \phi(t)dt \leq 0$ a contradiction. Therefore $\int_0^{p(f(x),f(x))} \phi(t)dt = 0$. it follows that $p(f(x), f(x)) = 0$. Since $p(f(x), f(x)) = 0$ and $p(f(x), f(f(x))) = 0$, yields $f(f(x)) = f(x)$.
Thus we have $g(f(x)) = f(f(x)) = f(x)$. Then $f(x)$ is a common fixed point of f and g.
We now prove the uniqueness of the common fixed point of f and g :
Suppose that there exist $x, y \in X$ such that $f(x) = g(x) = x$ and $f(y) = g(y) = y$. Assume $p(x, y) \neq 0$. Then, we have

$$\int_0^{p(x,y)} \phi(t)dt = \int_0^{p(f(x),f(y))} \phi(t)dt \leq (1-k_n) \int_0^{p(x,y)} \phi(t)dt$$

from which we have $\int_0^{p(x,y)} \phi(t)dt \leq 0$, which is a contradiction since p is nonnegative. Hence, by the assumption on ϕ, $\int_0^{p(x,y)} \phi(t)dt = 0$ which implies $p(x, y) = 0$.
In a similar manner, we also have $p(y, x) = 0$ and $p(x, x,) \leq p(x, y) + p(y, x)$; from which it follows that $p(x, x) = 0$.
Since $p(x, x) = 0$ and $p(x, y) = 0$; then $x = y$. □

Remark 2.1 *In proof of Theorem 2.1 of Boujari [5],* $p(f(x_0), f(x_1))$ *may be infinite and then we don't have* $lim_{n\to\infty} \int_0^{p(f(x_n)),f(x_{n+1}))} \phi(t)dt = 0$*, and one side of theorem is incorrect.*

Remark 2.2 *In Theorem 2.1 of Boujari [5], we can suppose that p-complete instead of d-complete that weaker and is beater result.*

The correct version of Theorem 2.2 of Boujari [5] is as follows:

Theorem 2.2 *Let (X, d) be a metric space, p be a w-distance on X. Suppose X is p-bounded and p-complete, and $f : X \to X$ a self-map. If there exists $c \in (0, 1)$ and map $g : X \to X$ which commutes with f such that $g(X) \subset f(X)$ and for each $x, y \in X$, satisfies*

$$\int_0^{p(g(x),g(y))} \varphi(t)dt \leq c \int_0^{p(f(x),f(y))} \varphi(t)dt, \tag{2.2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, nonnegative, and for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Indeed, f and g have a unique common fixed point.

It is necessary to mention that, the proof of the above theorem is similar to the proof of Theorem 2.1 of Boujari [5].

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