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Numerical solution of fuzzy Hunter-Saxton equation by using Adomian decomposition and Homotopy analysis methods

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Abstract

In this paper, a fuzzy Hunter-Saxton equation is solved by using the Adomian's decomposition method (ADM) and homotopy analysis method (HAM). The approximation solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

Key words: Hunter-Saxton equation; Adomian decomposition method; Homotopy analysis method; Generalized differentiability; Hukuhara-difference; Fuzzy number.

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1 Introduction

The Hunter-Saxton equation

$$u_{txx} = -2u_x u_{xx} - u u_{xxx},\ t > 0, \tag{1.1}$$

models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field, x being the space variable in a reference frame moving with the unperturbed wave speed and t being a slow time variable [12]. In recent years, some works have been done in order to find the numerical solution of this equation. For example [5,28,15,6,20,16,19,11,13,14]. In this work, we develop the ADM and HAM to solve the Eq.(1.1) with the fuzzy initial conditions as follows:

$$\begin{aligned} u(x,0) &= \widetilde{f}(x), \\ u_{tx}(a,t) &= \widetilde{g}(t), \\ u_t(a,t) &= \widetilde{g_1}(t). \end{aligned} \tag{1.2}$$

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1.1). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved in section 3. A example is presented in section 4 to illustrate the accuracy of these methods.

To obtain the approximate solution of Eq.(1.1), by integrating three times from Eq.(1.1) with respect to x, t and using the fuzzy initial conditions we obtain,

$$\begin{aligned} &\widetilde{u}(x,t) = \widetilde{F}(x,t) \oplus (-2) \odot \textstyle\int_0^t \int_a^x (x-t)\ \odot F_1(\widetilde{u}(x,t))\ dt\ dx \\ &\oplus(-1) \odot \textstyle\int_0^t \int_a^x (x-t)\ \odot F_2(\widetilde{u}(x,t))\ dt\ dx, \end{aligned} \tag{1.3}$$

where,

$$D^i(\tilde{u}(x,t)) = \frac{\partial^i \widetilde{u}(x,t)}{\partial x^i}, \quad i = 1,2,3,$$
$$\widetilde{F}(x,t) = \widetilde{f}(x) \oplus (x-a) \odot \int_0^t (\widetilde{g_1}(t) \oplus \widetilde{g}(t))\ dt,$$
$$F_1(u(x,t)) = D(\tilde{u}(x,t)) \odot D^2(\tilde{u}(x,t)),$$
$$F_2(u(x,t)) = \tilde{u}(x,t) \odot D^3(\tilde{u}(x,t)).$$

In Eq.(1.3), we assume $\widetilde{F}(x,t)$ is bounded for all t in $J = [0,T]$ and x in $[a,b]$ $(T,a,b \in \mathbb{R})$.

The terms $F_1(u(x,t)), F_2(u(x,t))$ are Lipschitz continuous with $D(F_i(u) - F_i(u^*)) \leq L_i\ D(u - u^*)(i = 1,2)$, and

$$\mid x - t \mid \leq M,$$

$$\alpha = T(b-a)M(2L_1 + L_2).$$

2 Definitions

The basic definitions of a fuzzy number are given in [1,2] as follows:

Definition 2.1 *A fuzzy number is a fuzzy set like* $u : \mathbb{R} \to [0,1]$ *which satisfies:*

1. u is an upper semi-continuous function,
2. $u(x) = 0$ *outside some interval [a,d],*
3. There are real numbers b, c *such as* $a \leq b \leq c \leq d$ *and*
3.1 $u(x)$ *is a monotonic increasing function on [a, b],*
3.2 $u(x)$ *is a monotonic decreasing function on [c, d],*
3.3 $u(x) = 1$ *for all* $x \in [b,c]$.

Definition 2.2 *A fuzzy number u in parametric form is a pair* $(\underline{u}, \overline{u})$ *of functions* $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, *which satisfy the following requirements:*

1. $\underline{u}(r)$ *is a bounded non-decreasing left continuous function in* $(0,1]$, *and right continuous at 0,*
2. $\overline{u}(r)$ *is a bounded non-increasing left continuous function in* $(0,1]$, *and right continuous at 0,*
3. $\underline{u}(r) \leq \overline{u}(r), \ 0 \leq r \leq 1.$

Definition 2.3 *for arbitrary* $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ *and* $\tilde{v} = (\underline{v}(r), \overline{v}(r))$ *and scalar* k*, we define addition, subtraction and scalar multiplication by* k *are respectively as following:*

$$\underline{u+v}(r) = \underline{u}(r) + \underline{v}(r), \qquad \overline{u+v}(r) = \overline{u}(r) + \overline{v}(r),$$

$$\underline{u-v}(r) = \underline{u}(r) - \overline{v}(r), \qquad \overline{u-v}(r) = \overline{u}(r) - \underline{v}(r),$$

$$\tilde{ku} = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0, \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0, \end{cases}$$

and if all of $u_i s$ *of* u *are positive we have:*

$$\underline{u} = \sum_{i=0}^{\infty} \underline{u}_i,$$

and

$$\overline{u} = \sum_{i=0}^{\infty} \overline{u}_i.$$

3 The fuzzy Hunter- Saxton equation

In this section, we are going to find solution of the Hunter- Saxton equation. If there is a fuzzy solution we find it. Let the solution of the Hunter- Saxton is positive, this hypothesis is correct because it's exact solution equation is e^{x-3t} and it is positive, then the u and u_x and u_{xx} are positive but by derivation of u to t the u_{txx} is negative. We can write the fuzzy haunter saxton equation

$$\tilde{u}_{txx} = (-2) \odot \tilde{u}_x \odot \tilde{u}_{xx} \oplus (-1) \odot \tilde{u}\tilde{u}_{xx}, \tag{3.1}$$

by interval arithmetics we can write the Hunter- Saxton equation in the following term:

$$[\underline{u}_{txx},\overline{u}_{txx}]=-2[\underline{u}_x,\overline{u}_x][\underline{u}_{xx},\overline{u}_{xx}]-[\underline{u},\overline{u}][\underline{u}_{xx},\overline{u}_{xx}], \tag{3.2}$$

by up hypothesis we can write two system by up and down and by positivity of u, u_{xx} and negativity of u_{txx} we have two following crisp systems:

$$\overline{u}(x,t)=\overline{F}(x,t)-2\int_0^t\int_a^x(x-t)\,F_1(\overline{u}(x,t))\,dt\,dx-\int_0^t\int_a^x(x-t)F_2(\overline{u}(x,t))\,dt\,dx, \tag{3.3}$$

$$\underline{u}(x,t)=\underline{F}(x,t)-2\int_0^t\int_a^x(x-t)\,F_1(\underline{u}(x,t))\,dt\,dx-\int_0^t\int_a^x(x-t)F_2(\underline{u}(x,t))\,dt\,dx, \tag{3.4}$$

now we can solve this two crisp equations.

4 The Iterative Methods

4.1 Description of ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu+Ru+Nu=g_1(x,t), \tag{4.1}$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(4.1), and using the given conditions we obtain

$$u(x,t)=f_1(x)-L^{-1}(Ru)-L^{-1}(Nu), \tag{4.2}$$

where the function $f_1(x)$ represents the terms arising from integrating the source term $g_1(x,t)$. The nonlinear operator $Nu=G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{4.3}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows [7,9,27]:

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}, \tag{4.4}$$

where,

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1'(u_0), \\ A_2 &= u_2 G_1'(u_0) + \tfrac{1}{2!} u_1^2 G_1''(u_0), \\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \tfrac{1}{3!} u_1^3 G_1'''(u_0), ... \end{aligned} \tag{4.5}$$

The standard Adomian decomposition technique represents the solution of $\overline{u}(x,t)$ and $\underline{u}(x,t)$ in Eq.(4.1) as the following series,

$$\overline{u}(x,t) = \sum_{i=0}^{\infty} \overline{u}_i(x,t), \tag{4.6}$$

$$\underline{u}(x,t) = \sum_{i=0}^{\infty} \underline{u}_i(x,t), \tag{4.7}$$

where, the components $\overline{u}_0, \overline{u}_1, \ldots$ and $\underline{u}_0, \underline{u}_1, \ldots$ are usually determined recursively by

$$\begin{aligned} \overline{u}_0 \quad &= \overline{F}(x,t), \\ \overline{u}_1 \quad &= -2\int_0^t \int_a^x (x-t)\overline{A}_0(x,t)\ dt\ ds - \int_0^t \int_a^x (x-t)\overline{B}_0(x,t)\ dt\ dx, \\ &\vdots \\ \overline{u}_{n+1} &= -2\int_0^t \int_a^x (x-t)\overline{A}_n(x,t)\ dt\ ds - \int_0^t \int_a^x (x-t)\overline{B}_n(x,t)\ dt\ dx, \quad n \geq 0, \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
\underline{u}_0 &= \underline{F}(x,t),\\
\underline{u}_1 &= -2\int_0^t\int_a^x (x-t)\underline{A}_0(x,t)\ dt\ ds - \int_0^t\int_a^x (x-t)\underline{B}_0(x,t)\ dt\ dx,\\
&\vdots\\
\overline{u}_{n+1} &= -2\int_0^t\int_a^x (x-t)\underline{A}_n(x,t)\ dt\ ds - \int_0^t\int_a^x (x-t)\underline{B}_n(x,t)\ dt\ dx, \quad n\geq 0,
\end{aligned} \tag{4.9}
$$

where the $\tilde{A}_n = [\underline{A}_n, \overline{A}_n]$, $n \geq 0$, are the Adomian polynomial determined by:

$$\overline{A}_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}(N\sum_{i=0}^{n}\lambda^i\overline{u}_i)]_{\lambda=0}$$

$$\underline{A}_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}(N\sum_{i=0}^{n}\lambda^i\underline{u}_i)]_{\lambda=0}$$

where,

$$
\begin{aligned}
\overline{A}_0 &= G_1(\overline{u}_0)\\
\overline{A}_1 &= \overline{u}_1 G_1'(\overline{u}_0)\\
\overline{A}_2 &= \overline{u}_2 G_1'(\overline{u}_0) + \frac{1}{2!}\overline{u}^2 G_1'(\overline{u}_0)\\
&\vdots
\end{aligned}
$$

and,

$$
\begin{aligned}
\underline{A}_0 &= G_1(\underline{u}_0)\\
\underline{A}_1 &= \underline{u}_1 G_1'(\underline{u}_0)\\
\underline{A}_2 &= \underline{u}_2 G_1'(\underline{u}_0) + \frac{1}{2!}\underline{u}^2 G_1'(\underline{u}_0)\\
&\vdots
\end{aligned}
$$

4.2 Description of the HAM

Consider

$$N[\overline{u}] = 0,$$

where N is a nonlinear operator, $\overline{u}(x,t)$ is unknown function and x is an independent variable. let $\overline{u}_0(x,t)$ denote an initial guess of the exact solution $\overline{u}(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - \overline{u}_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)]$$
$$= \hat{H}[\phi(x,t;q); \overline{u}_0(x,t), H_1(x,t), h, q]. \quad (4.10)$$

It should be emphasized that we have great freedom to choose the initial guess $\overline{u}_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$. Enforcing the homotopy Eq.(4.10) to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q); \overline{u}_0(x,t), H_1(x,t), h, q] = 0, \quad (4.11)$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - \overline{u}_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \quad (4.12)$$

When $q = 0$, the zero-order deformation Eq.(4.12) becomes

$$\phi(x;0) = \overline{u}_0(x,t), \quad (4.13)$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq.(4.12) is equivalent to

$$\phi(x,t;1) = \overline{u}(x,t). \quad (4.14)$$

Thus, according to Eq.(4.13) and Eq.(4.14), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $\overline{u}_0(x,t)$ to the exact solution $\overline{u}(x,t)$. Such a kind of continuous variation is called deformation in homotopy [3,4,17,18,21–23,10,24–26].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = \overline{u}_0(x,t) + \sum_{m=1}^{\infty} \overline{u}_m(x,t) q^m, \quad (4.15)$$

where,

$$\overline{u}_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\mid_{q=0} .$$

Let the initial guess $\overline{u}_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series Eq.(4.15) of $\phi(x,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$\overline{u}(x,t) = \phi(x,t;1) = \overline{u}_0(x,t) + \sum_{m=1}^{\infty} \overline{u}_m(x,t). \qquad (4.16)$$

From Eq.(4.15), we can write Eq.(4.12) as follows

$$\begin{aligned}(1-q)L[\phi(x,t,q) - \overline{u}_0(x,t)] &= (1-q)L[\textstyle\sum_{m=1}^{\infty} \overline{u}_m(x,t)\ q^m] \\ &= q\ h\ H_1(x,t)N[\phi(x,t,q)] \\ &\Rightarrow L[\textstyle\sum_{m=1}^{\infty} \overline{u}_m(x,t)\ q^m] \\ &- q\ L[\textstyle\sum_{m=1}^{\infty} \overline{u}_m(x,t)q^m] \\ &= q\ h\ H_1(x,t)N[\phi(x,t,q)]. \end{aligned} \qquad (4.17)$$

By differentiating Eq.(4.17) m times with respect to q, we obtain

$$\begin{aligned}&\{L[\textstyle\sum_{m=1}^{\infty} \overline{u}_m(x,t)\ q^m] - q\ L[\textstyle\sum_{m=1}^{\infty} \overline{u}_m(x,t)q^m]\}^{(m)} = \\ &\{q\ h\ H_1(x,t)N[\phi(x,t,q)]\}^{(m)} = \\ &m!\ L[\overline{u}_m(x,t) - \overline{u}_{m-1}(x,t)] = h\ H_1(x,t)\ m\ \tfrac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\mid_{q=0} .\end{aligned}$$

Therefore,

$$L[\overline{u}_m(x,t) - \chi_m \overline{u}_{m-1}(x,t)] = hH_1(x,t)\Re_m(\overline{u}_{m-1}(x,t)), \qquad (4.18)$$

where,

$$\Re_m(\overline{u}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\mid_{q=0}, \qquad (4.19)$$

and

$$\chi_m = \begin{cases} 0, \ m \leq 1, \\ 1, \ m > 1. \end{cases}$$

Note that the high-order deformation Eq.(4.18) is governing the linear operator L, and the term $\Re_m(\overline{u}_{m-1}(x,t))$ can be expressed simply by Eq.(4.19) for any nonlinear operator N.

To obtain the approximation solution of Eq.(1.3), according to HAM, let

$$N[\overline{u}(x,t)] = \overline{u}(x,t) - \overline{F}(x,t) + 2\int_0^t \int_a^x (x-t) F_1(\overline{u}(x,t)) \ dt \ dx +$$
$$\int_0^t \int_a^x (x-t) F_2(\overline{u}(x,t)) \ dt \ dx,$$

so,

$$\Re_m(\overline{u}_{m-1}(x,t)) = \overline{u}_{m-1}(x,t) - \overline{F}(x,t) + 2\int_0^t \int_a^x (x-t) F_1(\overline{u}(x,t))$$
$$dt \ dx + \int_0^t \int_a^x (x-t) F_2(\overline{u}(x,t)) \ dt \ dx. \tag{4.20}$$

Substituting Eq.(4.20) into Eq.(4.18)

$$L[\overline{u}_m(x,t) - \chi_m \overline{u}_{m-1}(x,t)] = hH_1(x,t)[\overline{u}_{m-1}(x,t) + 2\int_0^t \int_a^x (x-t)$$
$$F_1(\overline{u}(x,t)) \ dt \ dx + \int_0^t \int_a^x (x-t) F_2(\overline{u}(x,t)) \ dt \ dx + (1-\chi_m)\overline{F}(x,t)]. \tag{4.21}$$

We take an initial guess $\overline{u}_0(x,t) = \overline{F}(x,t)$, an auxiliary linear operator $L\overline{u} = \overline{u}$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into Eq.(4.21) to give the recurrence relation

$$\overline{u}_0(x,t) = \overline{F}(x,t),$$

$$\overline{u}_{n+1}(x,t) = -2\int_0^t \int_a^x (x-t) F_1(\overline{u}_n(x,t)) \ dt \ dx - \int_0^t \int_a^x (x-t)$$
$$F_2(\overline{u}_n(x,t)) \ dt \ dx, \quad n \geq 1. \tag{4.22}$$

Also, we can write

$$\underline{u}_0(x,t) = \underline{F}(x,t),$$

$$\underline{u}_{n+1}(x,t) = -2\int_0^t\int_a^x (x-t)F_1(\underline{u}_n(x,t))\ dt\ dx - \int_0^t\int_a^x (x-t)$$
$$F_2(\underline{u}_n(x,t))\ dt\ dx, \quad n \geq 1. \tag{4.23}$$

5 Existence and convergence of iterative methods

Theorem 5.1 *Let $0 < \alpha < 1$, then Eq.(1.3), has a unique solution.*
Proof. *Let u and u^* be two different solutions of Eq.(1.3) then*

$$D(\widetilde{u}, \widetilde{u}*) = D(\widetilde{F}(x,t) \oplus (-2) \odot \int_0^t\int_a^x (x-t)\ \odot F_1(\widetilde{u}(x,t))\ dt\ dx$$
$$\oplus(-1) \odot \int_0^t\int_a^x (x-t)\ \odot F_2(\widetilde{u}(x,t))\ dt\ dx,$$
$$\widetilde{F}(x,t) \oplus (-2) \odot \int_0^t\int_a^x (x-t)\ \odot F_1(\widetilde{u}^*(x,t))\ dt\ dx$$
$$\oplus(-1) \odot \int_0^t\int_a^x (x-t)\ \odot F_2(\widetilde{u}^*(x,t))\ dt\ dx)$$
$$\leq TM(b-a)(2L_1 + L_2)\ \mid u - u^* \mid = \alpha \mid u - u^* \mid .$$

From which we get $(1-\alpha)D(\widetilde{u}, \widetilde{u}^*) \leq 0$. Since $0 < \alpha < 1$, then $D(\widetilde{u}, \widetilde{u}^*) = 0$. Implies $\widetilde{u} = \widetilde{u}^*$ and completes the proof. □

Theorem 5.2 *The series solution $\overline{u}(x,t) = \sum_{i=0}^{\infty} \overline{u}_i(x,t)$ of Eq.(1.3) using ADM convergence when $0 < \alpha < 1$, $\mid \overline{u}_1(x,t) \mid < \infty$.*
Proof. *Define the sequence of partial sums $\overline{s}_n$, let $\overline{s}_n$ and $\overline{s}_m$ be arbitrary partial sums with $n \geq m$. We are going to prove that $\overline{s}_n$*

is a Cauchy sequence in this Banach space:

$$\begin{aligned}\| \overline{s}_n - \overline{s}_m \| &= max_{\forall x,t} \mid \overline{s}_n - \overline{s}_m \mid \\ &= max_{\forall x,t} \mid \textstyle\sum_{i=m+1}^{n} \overline{u}_i(x,t) \mid \\ &= max_{\forall x,t} \mid \textstyle\sum_{i=m+1}^{n}(-2\int_0^t \int_a^x (x-t)\overline{A}_{i-1}\ dt\ dx \\ &- \textstyle\int_0^t \int_a^x (x-t)\overline{B}_{i-1}\ dt\ dx \mid \\ &= max_{\forall x,t} \mid -2\textstyle\int_0^t \int_a^x (x-t)(\sum_{i=m}^{n-1}\overline{A}_i)\ dt\ dx \\ &- \textstyle\int_0^t \int_a^x (\sum_{i=m}^{n-1}\overline{B}_i)\ dt\ dx \mid .\end{aligned}$$

From [10], we have

$$\begin{aligned}\textstyle\sum_{i=m}^{n-1}\overline{A}_i &= F_1(\overline{s}_{n-1}) - F_1(\overline{s}_{m-1}), \\ \textstyle\sum_{i=m}^{n-1}\overline{B}_i &= F_2(\overline{s}_{n-1}) - F_2(\overline{s}_{m-1}).\end{aligned}$$

So,

$$\begin{aligned}\| \overline{s}_n - \overline{s}_m \| &= max_{\forall x,t} \mid -2\textstyle\int_0^t \int_a^x (x-t)[F_1(\overline{s}_{n-1}) - F_1(\overline{s}_{m-1})]\ dt\ dx \\ &- \textstyle\int_0^t \int_a^x (x-t)[F_2(\overline{s}_{n-1}) - F_2(\overline{s}_{m-1})]\ dt\ dx \mid \\ &\leq 2\textstyle\int_0^t \int_a^x (x-t) \mid F_1(\overline{s}_{n-1}) - F_1(\overline{s}_{m-1}) \mid\ dt\ dx \\ &+ \textstyle\int_0^t \int_a^x (x-t) \mid F_2(\overline{s}_{n-1}) - F_2(\overline{s}_{m-1}) \mid\ dt\ dx \\ &\leq \alpha \| \overline{s}_n - \overline{s}_m \| .\end{aligned}$$

Let $n = m+1$, *then*

$$\| \overline{s}_n - \overline{s}_m \| \leq \alpha \| \overline{s}_m - \overline{s}_{m-1} \| \leq \alpha^2 \| \overline{s}_{m-1} - \overline{s}_{m-2} \| \leq ... \leq \alpha^m \| \overline{s}_1 - \overline{s}_0 \| .$$

We have,

$$\begin{aligned}\| \overline{s}_n - \overline{s}_m \| &\leq \| \overline{s}_{m+1} - \overline{s}_m \| + \| \overline{s}_{m+2} - \overline{s}_{m+1} \| + ... + \| \overline{s}_n - \overline{s}_{n-1} \| \\ &\leq [\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}] \| \overline{s}_1 - \overline{s}_0 \| \\ &\leq \alpha^m[1 + \alpha + \alpha^2 + ... + \alpha^{n-m-1}] \| \overline{s}_1 - \overline{s}_0 \| \\ &\leq \alpha^m[\tfrac{1-\alpha^{n-m}}{1-\alpha}] \| \overline{u}_1(x,t) \| .\end{aligned}$$

Since $0 < \alpha < 1$, *we have* $(1 - \alpha^{n-m}) < 1$, *then*

$$\| \overline{s}_n - \overline{s}_m \| \leq \frac{\alpha^m}{1-\alpha} max_{\forall t} \mid \overline{u}_1(x,t) \mid . \qquad (5.1)$$

But $\mid \overline{u}_1(x,t) \mid < \infty$, *so, as* $m \to \infty$, *then* $\| \overline{s}_n - \overline{s}_m \| \to 0$. *We conclude that* s_n *is a Cauchy sequence , therefore*

$$\overline{u}(x,t) = \lim_{n\to\infty} \overline{u}_n(x,t).$$

Also, we can write,

$$\underline{u}(x,t) = \lim_{n\to\infty} \underline{u}_n(x,t).$$

Therefore,

$$\widetilde{u}(x,t) = \lim_{n\to\infty} \widetilde{u}_n(x,t). \qquad \square$$

Theorem 5.3 *If the series solutions Eq.(4.22) and Eq.(4.23) of Eq.(1.3) using HAM convergent then it converges to the exact solution of the Eq.(1.3).*
Proof. *We assume:*

$$\phi_{k+1}(x,t) = F(x,t) \oplus$$
$$\textstyle\sum_{i=1}^{k+1} [(-2) \odot \int_0^t \int_a^x (x-t) \ \odot F_1(u_i(x,t)) \ dt \ dx$$
$$\oplus (-1) \odot \textstyle\int_0^t \int_a^x (x-t) \ \odot F_2(u_i(x,t)) \ dt \ dx], \quad k \geq 0.$$

$D(\phi_{k+1}(x,t),\phi_k(x,t)) = D(F(x,t)\oplus$
$\sum_{i=1}^{k+1}[(-2)\odot\int_0^t\int_a^x(x-t)\ \odot F_1(u_i(x,t))\ dt\ dx$
$\oplus(-1)\odot\int_0^t\int_a^x(x-t)\ \odot F_2(u_i(x,t))\ dt\ dx], F(x,t)\oplus$
$\sum_{i=1}^{k+1}[(-2)\odot\int_0^t\int_a^x(x-t)\ \odot F_1(u_{i-1}(x,t))\ dt\ dx$
$\oplus(-1)\odot\int_0^t\int_a^x(x-t)\ \odot F_2(u_{i-1}(x,t))\ dt\ dx],) =$
$D(\phi_k(x,t)\oplus(-2)\odot\int_0^t\int_a^x(x-t)\ \odot F_1(u_k(x,t))\ dt\ dx$
$\oplus(-1)\odot\int_0^t\int_a^x(x-t)\ \odot F_2(u_k(x,t))\ dt\ dx, \phi_k(x,t))$
$= D((-2)\odot\int_0^t\int_a^x(x-t)\ \odot F_1(u_k(x,t))\ dt\ dx$
$\oplus(-1)\odot\int_0^t\int_a^x(x-t)\ \odot F_2(u_k(x,t))\ dt\ dx, \widetilde{0}) \leq D(u_k(x,t),\widetilde{0})$

$D(u_k(x,t),\widetilde{0}) \leq \alpha^k D(F,\widetilde{0})$
$\Longrightarrow D(\phi_{k+1}(x,t),\phi_k(x,t)) \leq \alpha^{k+1}D(F,\widetilde{0})$
$\Longrightarrow \sum_{k=0}^{\infty} D(\phi_{k+1}(x,t),\phi_k(x,t)) \leq \alpha^{k+1}D(F,\widetilde{0})\sum_{k=0}^{\infty}\alpha^k.$

6 Numerical example

In this section, we compute a numerical example which is solved by the ADM and HAM. The program has been provided with Mathematica 6.

Lemma 6.1 *The computational complexity of the ADM is $O(n^3)$ and HAM is $O(n)$.*
Proof. *The number of computations including division, production, sum and subtraction.*
ADM:
$\overline{A}_n, \overline{B}_n, \underline{A}_n, \underline{B}_n$:
$2n^2+10n+8.$
$\overline{u}_0, \underline{u}_0 : 8.$
$\overline{u}_1, \underline{u}_1 : 48.$

$\vdots$

$\overline{u}_{n+1}, \underline{u}_{n+1} : 48.$

The total number of the computations is equal to

$\sum_{i=0}^{n+1} \overline{u}_i(x,t) + \sum_{i=0}^{n+1} \underline{u}_i(x,t) = O(n^3).$

HAM:

$\overline{u}_0, \underline{u}_0 : 8.$

$\overline{u}_1, \underline{u}_1 : 16.$

$\vdots$

$\overline{u}_{n+1}, \underline{u}_{n+1} : 16.$

The total number of the computations is equal to

$\sum_{i=0}^{n+1} \overline{u}_i(x,t) + \sum_{i=0}^{n+1}$

$\underline{u}_i(x,t) = O(n).$

By comparing the results of computational complexity, we see that the number of computations in HAM is less than the number of computations in ADM.

Example 6.1 *Consider the fuzzy hunter-Saxeton equation as follows:*

$$\widetilde{u}_t \oplus 2 \odot \widetilde{u}_x \oplus (-1) \odot \widetilde{u}_{xxt} \oplus \widetilde{u} \odot \widetilde{u}_x = 2 \odot \widetilde{u}_x \odot \widetilde{u}_x x \oplus \widetilde{u} \odot \widetilde{u}_x xx. \quad (6.1)$$

The exact solution is

$$u(x,t) = (\underline{u}(x,t,\gamma), \overline{u}(x,t,\gamma) = (e^{x-3t}(-\gamma^2+3\gamma-2), e^{x-3t}(\gamma^2-3\gamma+2)).$$

Table 1 Numerical results for Example 1

(x,t)	Errors(D)(ADM, n=12))	Errors(D)(HAM,n=5)
(0.3,0.15)	0.050281	0.030281
(0.35,0.20)	0.054184	0.032267
(0.4,.25)	0.058754	0.036754
(0.45,0.30)	0.062683	0.038867
(0.5, 0.37)	0.065375	0.043578
(0.55,0.4 0)	0.067284	0.045638
(0.6, 0.45)	0.069881	0.047245
(0.65,0.48)	0.072674	0.051257
(0.7,0.5 0)	0.075843	0.053897
(0.75, 0.54)	0.077698	0.056245
(0.8, 0.62)	0.079675	0.057895

7 Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate analytical solution of the fuzzy Hunter-Saxton equation. For this purpose, we showed that the HAM is more rapid convergence than the ADM. Also, the number of computations in HAM is less than the number of computations in ADM.

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