



Convergence Theorems for α -Nonexpansive Mappings in $CAT(0)$ Spaces

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Abstract

In this paper we derive convergence theorems for an α -nonexpansive mapping of a nonempty closed and convex subset of a complete $CAT(0)$ space for SP-iterative process and Thianwan's iterative process.

Key words: $CAT(0)$ spaces, α -Nonexpansive mapping, Δ -convergence, SP-iteration, Thianwan's iteration.

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1 Introduction

The purpose of this paper is to study fixed point theorems for α -nonexpansive mappings in $CAT(0)$ spaces. A metric space X is a $CAT(0)$

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space (see Bridson and Haefliger [2]) if it is geodesically connected and if every geodesic triangle in X is at least as ‘thin’ as its comparison triangle in the Euclidean plane. Our approach is to prove Δ -convergence theorems for SP-iteration and Thianwan’s iterations for α -nonexpansive mappings in $CAT(0)$ spaces.

Here are the details. Let X be a $CAT(0)$ space and let C be a nonempty subset of X and $T : C \rightarrow X$ be a mapping. Denote $F(T)$ by the set of fixed points of T , i.e., $F(T) = \{x \in C : Tx = x\}$.

Definition 1.1 T is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$ and that T is quasi-nonexpansive if $F(T) \neq \phi$ and $d(Tx, y) \leq d(x, y)$ for all $x \in C$ and $y \in F(T)$.

In 2011, Aoyama and Kohsaka [1] defined α -non-expansive mappings in Banach spaces. We introduce the notion of this mapping in $CAT(0)$ spaces.

Definition 1.2 A mapping $T : C \rightarrow X$ is said to be an α -non-expansive for some real number $\alpha < 1$ if $d(Tx, Ty)^2 \leq \alpha d(Tx, y)^2 + \alpha d(Ty, x)^2 + (1 - 2\alpha)d(x, y)^2$ for all $x, y \in C$.

Clearly, 0-non-expansive maps are exactly non-expansive maps.

Definition 1.3 ([6]) Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X . For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n)$. The asymptotic $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.

Remark 1.4 $A(\{x_n\})$ consists of exactly one point in $CAT(0)$ spaces (see, e.g., [5, Proposition 7]).

Definition 1.5 ([6]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim x_n = x$ and call x the Δ -limit of $\{x_n\}$.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y . When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane \mathbb{E}^2 such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a $CAT(0)$ space if all geodesic triangles satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ . Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \quad (\text{CN})$$

This is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfy (CN) inequality.

Now we collect some results which are used in our main results:

Lemma 1.6 ([6]) *Let (X, d) be a $CAT(0)$ space. Then for $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \quad (1.1)$$

Lemma 1.7 ([6]) *Let X be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z) \quad (1.2)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 1.8 ([6]) *Let (X, d) be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2 \quad (1.3)$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

Lemma 1.9 ([7]) *Every bounded sequence in a complete $CAT(0)$ space always has a Δ -convergent subsequence.*

Lemma 1.10 ([4]) *If C is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C .*

In 2009, Thianwan [12] introduced an iteration named by his name, i.e., Thianwan's iteration which is defined by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n) y_n, \\ y_n &= \beta_n T x_n + (1 - \beta_n) x_n, \quad \text{for all } n \geq 1, \end{aligned} \quad (1.4)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

In 2011, Phuengrattana and Suantai [10] defined the SP-iteration as follows:

$$\begin{aligned} x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n) y_n, \\ y_n &= \beta_n T z_n + (1 - \beta_n) z_n, \\ z_n &= \gamma_n T x_n + (1 - \gamma_n) x_n, \end{aligned} \quad (1.5)$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in $[0, 1]$. They showed that the Mann, Ishikawa and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and non-decreasing functions. Clearly Thianwan's iteration and Mann iterations are special cases of the SP-iteration.

Example 1.11 Here we provide an example which proves that the SP-iteration converges much faster than the other iterations, for the increasing function $f(x) = 2x^3 - 7x^2 + 8x - 2$, the Ishikawa iteration converges in 22nd iterations while the SP-iteration converges in 2nd iteration with $x_0 = 0.8$.

Recall that SP-iteration and Thianwan's iteration in $CAT(0)$ spaces are described as follows:

For any initial point x_1 in C , we define the iterates $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \alpha_n T y_n \oplus (1 - \alpha_n) y_n \\ y_n = \beta_n T x_n \oplus (1 - \beta_n) x_n \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$. This iteration is Thianwan's new two step iteration. And if the iterates $\{x_n\}$ is defined as

$$\begin{cases} x_{n+1} = \alpha_n T y_n \oplus (1 - \alpha_n) y_n, \\ y_n = \beta_n T z_n \oplus (1 - \beta_n) z_n, \\ z_n = \gamma_n T x_n \oplus (1 - \gamma_n) x_n, \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in $[0, 1]$. Then it is known as SP-iteration.

Remark 1.12 If $\gamma_n = 0$ then (1.5) reduces to the iterative process (1.4).

2 Main Results

Now we are all set to prove our main results. We start with proving key lemmas for later use.

Lemma 2.1 ([9]) Let C be a nonempty subset of a $CAT(0)$ space X . Let $T : C \rightarrow X$ be an α -nonexpansive mapping for some real number $\alpha < 1$ such that $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.

Proof. Let $x \in C$ and $z \in F(T)$. Then we have

$$\begin{aligned}
d(Tx, z)^2 &= d(Tx, Tz)^2 \\
&\leq \alpha d(Tx, z)^2 + \alpha d(x, Tz)^2 + (1 - 2\alpha)d(x, z)^2 \\
&= \alpha d(Tx, z)^2 + \alpha d(x, z)^2 + (1 - 2\alpha)d(x, z)^2 \\
&= \alpha d(Tx, z)^2 + (1 - \alpha)d(x, z)^2.
\end{aligned}$$

Therefore, $d(Tx, z) \leq d(x, z)$. This inequality ensures the closedness of $F(T)$.

Lemma 2.2 ([9]) *Let C be a nonempty closed and convex subset of a $CAT(0)$ space X . Let $T : C \rightarrow X$ be an α -nonexpansive mapping for some $\alpha < 1$. Then the following conditions hold:*

(i) *If $0 \leq \alpha < 1$, then*

$$\begin{aligned}
d(x, Ty)^2 &\leq \frac{1 + \alpha}{1 - \alpha} d(x, Tx)^2 \\
&\quad + \frac{2}{1 - \alpha} \{ \alpha d(x, y) + d(Tx, Ty) \} d(x, Tx) + d(x, y)^2,
\end{aligned}$$

for all $x, y \in C$.

(ii) *If $\alpha < 0$, then*

$$\begin{aligned}
d(x, Ty)^2 &\leq d(x, Tx)^2 \\
&\quad + \frac{2}{1 - \alpha} \{ (-\alpha) d(Tx, y) + d(Tx, Ty) \} d(x, Tx) + d(x, y)^2,
\end{aligned}$$

for all $x, y \in C$.

Proof. (i) Observe

$$\begin{aligned}
d(x, Ty)^2 &\leq [d(x, Tx) + d(Tx, Ty)]^2 \\
&= d(x, Tx)^2 + d(Tx, Ty)^2 + 2d(x, Tx)d(Tx, Ty) \\
&\leq d(x, Tx)^2 + \alpha d(Tx, y)^2 + \alpha d(x, Ty)^2 + (1 - 2\alpha)d(x, y)^2 \\
&\quad + 2d(x, Tx)d(Tx, Ty) \\
&\leq d(x, Tx)^2 + \alpha[d(Tx, x) + d(x, y)]^2 + \alpha d(x, Ty)^2 \\
&\quad + (1 - 2\alpha)d(x, y)^2 + 2d(x, Tx)d(Tx, Ty) \\
\\
&\leq d(x, Tx)^2 + \alpha d(Tx, x)^2 + \alpha d(x, y)^2 + 2\alpha d(Tx, x)d(x, y) \\
&\quad + \alpha d(x, Ty)^2 + (1 - 2\alpha)d(x, y)^2 + 2d(x, Tx)d(Tx, Ty) \\
&= (1 + \alpha)d(x, Tx)^2 + 2\alpha d(Tx, x)d(x, y) + \alpha d(x, Ty)^2 \\
&\quad + (1 - \alpha)d(x, y)^2 + 2d(x, Tx)d(Tx, Ty)
\end{aligned}$$

This implies that

$$\begin{aligned}
d(x, Ty)^2 &\leq \frac{1 + \alpha}{1 - \frac{2}{\alpha}\alpha} d(x, Tx)^2 \\
&\quad + \frac{2}{1 - \alpha} \{\alpha d(x, y) + d(Tx, Ty)\} d(x, Tx) \\
&\quad + d(x, y)^2
\end{aligned}$$

(ii) Observe

$$\begin{aligned}
d(x, Ty)^2 &\leq [d(x, Tx) + d(Tx, Ty)]^2 \\
&= d(x, Tx)^2 + d(Tx, Ty)^2 + 2d(x, Tx)d(Tx, Ty) \\
&\leq d(x, Tx)^2 + \alpha d(Tx, y)^2 \\
&+ \alpha d(x, Ty)^2 + (1 - 2\alpha)d(x, y)^2 \\
&+ 2d(x, Tx)d(Tx, Ty) \\
&= d(x, Tx)^2 + \alpha d(Tx, y)^2 + \alpha d(x, Ty)^2 \\
&+ (1 - \alpha)d(x, y)^2 \\
&- \alpha d(x, y)^2 + 2d(x, Tx)d(Tx, Ty) \\
&\leq d(x, Tx)^2 + \alpha d(Tx, y)^2 + \alpha d(x, Ty)^2 \\
&+ (1 - \alpha)d(x, y)^2 \\
&- \alpha[d(x, Tx)^2 + d(Tx, y)^2 + 2d(x, Tx)d(Tx, y)] \\
&+ 2d(x, Tx)d(Tx, Ty) \\
&= (1 - \alpha)d(x, Tx)^2 + \alpha d(x, Ty)^2 \\
&+ (1 - \alpha)d(x, y)^2 - 2\alpha d(x, Tx)d(Tx, y) \\
&+ 2d(x, Tx)d(Tx, Ty) \\
&= (1 - \alpha)d(x, Tx)^2 + \alpha d(x, Ty)^2 \\
&+ (1 - \alpha)d(x, y)^2 \\
&+ 2[(-\alpha)d(Tx, y) + d(Tx, Ty)]d(x, Tx)
\end{aligned}$$

This implies that

$$\begin{aligned}
d(x, Ty)^2 &\leq d(x, Tx)^2 + \frac{2}{1 - \alpha}[(-\alpha)d(Tx, y) \\
&+ d(Tx, Ty)]d(x, Tx) + d(x, y)^2.
\end{aligned}$$

Lemma 2.3 *Let C be a nonempty closed and convex subset of a complete $CAT(0)$ space X and $T : C \rightarrow C$ be an α -nonexpansive mapping for*

some $\alpha < 1$. If $\{x_n\}$ is a sequence in C such that $d(Tx_n, x_n) \rightarrow 0$ and $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = z$ for some z in X then $z \in C$ and $Tz = z$.

Proof. It follows from Lemma 1.10 that $z \in C$. Let $0 \leq \alpha < 1$. By Lemma 2.2(i), we deduce that

$$\begin{aligned} d(x_n, Tz)^2 &\leq \frac{1+\alpha}{1-\alpha} d(x_n, Tx_n)^2 \\ &\quad + \frac{2}{1-\alpha} \{\alpha d(x_n, z) + d(Tx_n, Tz)\} d(x_n, Tx_n) + d(x_n, z)^2 \end{aligned}$$

for all n in N . Thus we have $\limsup_{n \rightarrow \infty} d(x_n, Tz) \leq \limsup_{n \rightarrow \infty} d(x_n, z)$. Let $\alpha < 0$. Then by Lemma 2.2(ii), we have

$$\begin{aligned} d(x_n, Tz)^2 &\leq d(x_n, Tx_n)^2 \\ &\quad + \frac{2}{1-\alpha} \{(-\alpha) d(Tx_n, z) + d(Tx_n, Tz)\} d(x_n, Tx_n) + d(x_n, z)^2 \end{aligned}$$

for all n in N . This implies again that

$$\limsup_{n \rightarrow \infty} d(x_n, Tz) \leq \limsup_{n \rightarrow \infty} d(x_n, z).$$

By the uniqueness of asymptotic centers, $Tz = z$.

Lemma 2.4 *Let C be a nonempty closed and convex subset of a $CAT(0)$ space X and $T : C \rightarrow C$ be an α -nonexpansive mapping for some $\alpha < 1$. Let a sequence $\{x_n\}$ with x_1 in C be defined by (1.5) such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are arbitrary sequences of positive numbers in $[0, 1]$. Let $p \in F(T)$. Then the following assertions hold:*

- (i) $\max\{d(x_{n+1}, p), d(y_n, p)\} \leq d(x_n, p)$ for $n = 1, 2, \dots$.
- (ii) $\lim_{n \rightarrow \infty} d(x_n, p)$ exists.
- (iii) $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists.

Proof. Consider

$$\begin{aligned}
d(y_n, p) &= d(\beta_n Tz_n \oplus (1 - \beta_n)z_n, p) \\
&\leq \beta_n d(Tz_n, p) + (1 - \beta_n)d(z_n, p) \\
&\leq \beta_n d(z_n, p) + (1 - \beta_n)d(z_n, p) \\
&= d(z_n, p)
\end{aligned}$$

$$\begin{aligned}
&= d(\gamma_n Tx_n \oplus (1 - \gamma_n)x_n, p) \\
&\leq \gamma_n d(Tx_n, p) + (1 - \gamma_n)d(x_n, p) \\
&\leq \gamma_n d(x_n, p) + (1 - \gamma_n)d(x_n, p) \\
&= d(x_n, p).
\end{aligned}$$

Consequently,

$$\begin{aligned}
d(x_{n+1}, p) &= d(\alpha_n Ty_n \oplus (1 - \alpha_n)y_n, p) \\
&\leq \alpha_n d(Ty_n, p) + (1 - \alpha_n)d(y_n, p) \\
&\leq \alpha_n d(y_n, p) + (1 - \alpha_n)d(y_n, p) \\
&= d(y_n, p) \\
&\leq d(x_n, p).
\end{aligned}$$

This implies that $\{d(x_n, p)\}$ is a bounded and non-increasing sequence. Thus $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. In the same manner, we see that $\{d(x_n, F(T))\}$ is also a bounded non-increasing real sequence and thus converges.

Lemma 2.5 *Let C be a nonempty closed and convex subset of a CAT(0) space X and $T : C \rightarrow C$ be an α -nonexpansive mapping for some $\alpha < 1$. Let a sequence $\{x_n\}$ with x_1 in C be defined by (1.5) such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are arbitrary sequences of positive numbers in $[0, 1]$. Let $p \in F(T)$. Then $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$.*

Proof. From the above Lemma 2.4(iii), $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists. Let $p \in F(T)$ and

$$\lim_{n \rightarrow \infty} d(x_n, p) = c. \quad (2.1)$$

Now we prove that $\lim_{n \rightarrow \infty} d(y_n, p) = c$.

Since $d(x_{n+1}, p) \leq d(y_n, p)$. This implies

$$\lim_{n \rightarrow \infty} d(x_{n+1}, p) \leq \lim_{n \rightarrow \infty} d(y_n, p) \quad \text{or} \quad c \leq \lim_{n \rightarrow \infty} d(y_n, p). \quad (2.2)$$

But $d(y_n, p) \leq d(x_n, p)$. This implies

$$\lim_{n \rightarrow \infty} \sup d(y_n, p) \leq c. \quad (2.3)$$

From (2.2) and (2.3), we get

$$\lim_{n \rightarrow \infty} d(y_n, p) = c. \quad (2.4)$$

Similarly we can get

$$\lim_{n \rightarrow \infty} d(z_n, p) = c. \quad (2.5)$$

Now

$$\begin{aligned} d(z_n, p)^2 &= d(\gamma_n T x_n \oplus (1 - \gamma_n)x_n, p)^2 \\ &\leq \gamma_n d(T x_n, p)^2 + (1 - \gamma_n) d(x_n, p)^2 - \gamma_n(1 - \gamma_n) d(T x_n, x_n)^2 \\ &\leq d(x_n, p)^2 - \gamma_n(1 - \gamma_n) d(T x_n, x_n)^2. \end{aligned}$$

Thus

$$\gamma_n(1 - \gamma_n) d(T x_n, x_n)^2 \leq d(x_n, p)^2 - d(z_n, p)^2,$$

so that

$$d(T x_n, x_n)^2 \leq \frac{1}{\gamma_n(1 - \gamma_n)} [d(x_n, p)^2 - d(z_n, p)^2].$$

By (2.1) and (2.5), $\limsup d(T x_n, x_n) \leq 0$ and hence $\lim_{n \rightarrow \infty} d(T x_n, x_n) = 0$.

Theorem 2.6 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \rightarrow C$ be an α -nonexpansive mapping with*

$F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (1.5). Then $\{x_n\}$ Δ -converges to a fixed point of T .

Proof. It follows from Lemma 2.5 that $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$. We now let $\omega_w(x_n) = \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subseteq F(T)$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 1.9, 1.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. Since $\lim_n d(v_n, Tv_n) = 0$, then $v \in F(T)$ by Lemma 2.3. We claim that $u = v$. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_n d(v_n, v) &< \limsup_n d(v_n, u) \\ &\leq \limsup_n d(u_n, u) \\ &= \limsup_n d(u_n, v) \\ &= \limsup_n d(x_n, v) \\ &= \limsup_n d(v_n, v), \end{aligned}$$

which is a contradiction and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ -converges to a fixed point of T , it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 1.9, 1.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in F(T)$. We can complete the proof by showing that $x = v$. If not, since $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_n d(v_n, v) &< \limsup_n d(v_n, v) \\ &\leq \limsup_n d(x_n, x) \\ &< \limsup_n d(x_n, v) \\ &\leq \limsup_n d(v_n, x) \end{aligned}$$

which is a contradiction and hence the conclusion follows.

Remark 2.7 *As we know that Thianwan's iteration is the special case for SP-iteration for $\gamma_n = 0$. So, the above Δ -convergence result is also hold for Thianwan's iterative process.*

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