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On the singular fuzzy linear system of equations

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Abstract

The linear system of equations $A\tilde{x} = \tilde{b}$ where $A = [a_{ij}] \in C^{n\times n}$ is a crisp singular matrix and the right-hand side is a fuzzy vector is called a singular fuzzy linear system of equations. In this paper, solving singular fuzzy linear systems of equations using generalized inverses such as Drazin inverse and pseudo-inverse are investigated.

Key words: Drazin inverse, Singular fuzzy linear system, Minimal solution, Singular matrices.

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1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh Dubois and Prade. The importance of the introduced notion of fuzzy set was realized and has successfully been applied in almost all the branches of science and technology. Recently fuzzy set theory has been applied in pure mathematices by Tripathy and Baruah [1], Tripathy and Borgohain [2], Tripathy, Sen and Nath [3], Tripathy and Das [4], Tripathy and Sarma [5], Tripathy, Baruah, Et and Gungor [6], Tripathy and Ray [7] and many others. We refer the reader to [8,9,10] for more information on fuzzy numbers and fuzzy arithmetic.

Solving linear system of equations when the coefficient matrix is a crisp matrix and the right-hand side is a fuzzy vector have been studied by many authors [11,12,13,14]. Friedman et al. [9] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy number vector. Prof. S.Abbasbandy et al. proposed a method for finding minimal solution of general dual fuzzy linear systems [15]. In [16] proposed a model to solve fuzzy linear system $Ax = b$, wherein $A \in C^{n\times n}$ is a nonsingular crisp matrix using ordinary inverse. In this method the original system with matrix A is replaced by two $n \times n$ crisp linear system. Ezzati give a method for solving fuzzy linear system [17]. Normal equations for singular fuzzy linear systems is given [18]. In this paper, Ezzati's method is extended and on the singular fuzzy linear systems is performed.

The consistent singular fuzzy linear system of equations has a set solution and the inconsistent singular fuzzy linear system of equations has a least squares set solution. For any matrix $A \in C^{n\times n}$, even singular matrices, index and Drazin inverse of A exists and is unique[19,20]. In section 2, we recall some preliminaries for index of matrix, Drazin inverse and pseudoinverse. In section 3, some new results on singular matrices is given. A new method for solving singular fuzzy linear system is proposed in section 4. Then we give numerical examples to illustrate previous sections in section 5. Section 6 ends the paper with the conclusions and suggestions remarks.

2 Preliminaries and Basic Definitions

In this section, the properties of the Drazin inverse and index of matrix $A \in C^{n\times n}$ are needed. For this reason, we start our study by introducing the concept of index of matrix, Drazin inverse, pseudoinverse and application of them in solving linear system of equations. Also give some definitions on the fuzzy numbers and fuzzy linear system of equations. We refer the reader to [19,21,22,23,24,25].

Definition 2.1 *Let $A \in C^{n\times n}$. The index of matrix A is equivalent to the dimension of largest Jordan block corresponding to the zero eigenvalue of A and is denoted by $ind(A)$.*

Some properties of index of matrix are listed below. The details are explained in [26].
1. $A \in C^{n\times n}, \quad o \leq ind(A) \leq n,$
2. $ind(A) = ind(A^T),$
3. $det(A) = 0 \iff ind(A) \neq 0.$

Definition 2.2 *Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A, denoted by A^D, if X satisfies the following conditions*

$$AX = XA,\ XAX = X,\ A^kXA = A^k.$$

When $ind(A) = 1$, A^D is called the group inverse of A, and denoted by A_g.

Theorem 2.1 [19, 25] *Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. We may assume that the Jordan normal form of A has the form as follows*

$$A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1},$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r, and N is a nilpotent matrix that $N^k = \bar{o}$. Then we can write the Drazin *inverse of A in the form*

$$A^D = P \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

When $ind(A) = 1$, obviously, $N = \bar{o}$.

Theorem 2.2 [26] *For any matrix $A \in C^{n\times n}$ the index and Drazin inverse of A exists and is unique.*

Theorem 2.3 [19] *$A^D b$ is a solution of*

$$Ax = b, k = ind(A), \tag{2.1}$$

if and only if $b \in R(A^k)$, and $A^D b$ is an unique solution of (2.1) provided that $x \in R(A^k)$.

Definition 2.3 *Let $A \in C^{m\times n}$. The matrix X of order $n \times m$ is the pseudoinverse of A ,denoted by A^+ , if X satisfies the following conditions*

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

Theorem 2.4 [27] *Let $A \in C^{m\times n}$. We may assume that the singular value decomposition of A has the form as follows*

$$A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} Q^*,$$

where P is an $m \times m$ unitary matrix, D is an $m \times n$ diagonal matrix, and Q is an $n \times n$ unitary matrix, then putting

$$A^+ = Q^* \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^*.$$

Theorem 2.5 [20] *Any consistent singular linear system of equations, is equivalent to an full-rank underdetermined linear system.*

Definition 2.4 *The set of all these fuzzy numbers in parametric form is denoted by E. A fuzzy number $\widetilde{u}$ in parametric form is a pair $(\bar{u}(r), \underline{u}(r))$ of functions $\bar{u}(r), \underline{u}(r), 0 \leq r \leq 1$, which satisfy the following requirements*
1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\bar{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$.

Definition 2.5 *For arbitrary fuzzy numbers $\widetilde{x} = (\underline{x}(r), \bar{x}(r))$, $\widetilde{y} = (\underline{y}(r), \bar{y}(r))$ and $k \in R$, we may define the addition and the scalar multiplication of fuzzy numbers as*
1. $\widetilde{x} + \widetilde{y} = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$,
2. $k \times \widetilde{x} = \begin{cases} (k\underline{x}(r), k\bar{x}(r)) \ k \geq 0 \\ (k\bar{x}(r), k\underline{x}(r)) \ k < 0 \end{cases}$

Definition 2.6 *The fuzzy linear system*

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}, \tag{2.2}$$

where $A = (a_{ij})$, $1 \leq i \leq n$ and $1 \leq j \leq n$ is a crisp singular matrix, and the element $\tilde{b}_{ij}$ in the right-hand side matrix are fuzzy numbers is called a singular fuzzy linear system. The fuzzy linear system (2.2) can be extended into a crisp linear system as follows

$$\begin{bmatrix} s_{1,1} & \cdots & s_{1,2n} \\ \vdots & \ddots & \vdots \\ s_{2n,1} & \cdots & s_{2n,2n} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \\ -\bar{b}_1 \\ \vdots \\ -\bar{b}_n \end{bmatrix},$$

where s_{ij} are determined as follows :

$$\begin{bmatrix} a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij}, s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij}, s_{i+n,j} = -a_{ij}, \end{bmatrix} \tag{2.3}$$

and any s_{ij} which is not determined by (2.3) is zero. Using matrix notation we get

$$SX = Y. \tag{2.4}$$

The structure of $S = (s_{ij}), 1 \leq i \leq 2n$ and $1 \leq j \leq 2n$ implies that and that

$$S = \begin{bmatrix} B & C \\ C & B \end{bmatrix},$$

where B contains the positive entries of A and C contains the absolute value of the negative entries of A , i.e., $A = B - C$.

For solving the fuzzy linear system (2.2) wherein $A \in C^{n\times n}$ is a non-singular crisp matrix. Ezzati [17] first solve the following system

$$\begin{cases} a_{11}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{1n}(\underline{x}_n + \bar{x}_n) = (\underline{y}_1 + \bar{y}_1), \\ a_{21}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{2n}(\underline{x}_n + \bar{x}_n) = (\underline{y}_2 + \bar{y}_2), \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{nn}(\underline{x}_n + \bar{x}_n) = (\underline{y}_n + \bar{y}_n), \end{cases}$$

and suppose the solution of this system is as

$$d = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} \underline{x}_1 + \bar{x}_1 \\ \vdots \\ \underline{x}_n + \bar{x}_n \end{bmatrix}.$$

Let matrices B and C have defined as definition 2.6. Now using matrix notation for (2.2), He get $A\tilde{x} = \tilde{y}$ or $(B - C)\tilde{x} = \tilde{y}$ and in parametric form $(B - C)(\underline{x}(r), \bar{x}(r)) = (\underline{y}(r), \bar{y}(r))$. Then he write this system as follows:

$$\begin{cases} B\underline{x}(r) - C\bar{x}(r) = \underline{y}(r), \\ B\bar{x}(r) - C\underline{x}(r) = \bar{y}(r), \end{cases}$$

are equivalent. By substituting of $\bar{x}(r) = d - \underline{x}(r)$ and $\underline{x}(r) = d - \bar{x}(r)$ in the first and second equation of above system, respectively. He get

$$(B + C)\underline{x}(r) = \underline{y}(r) + Cd,$$

and

$$(B + C)\bar{x}(r) = \bar{y}(r) + Cd.$$

If the ordinary inverse of matrix $F = B + C$ exist then, He can solve fuzzy linear system (2.2) by solving following crisp linear systems

$$\begin{cases} \underline{x}(r) = F^{-1}(\underline{y}(r) + Cd), \\ \bar{x}(r) = F^{-1}(\bar{y}(r) + Cd). \end{cases}$$

Definition 2.7 [27] *Consider a system of equations written in matrix form as $Ax = b$ where A is $m \times n$, x is $n \times 1$, and b is $m \times 1$. The minimal solution of this problem is defined as follows:*

1. If the system is consistent and has a unique solution, x, then the minimal solution is defined to be x.

2. If the system is consistent and has a set of solutions, then the minimal solution is the element of this set having the least Euclidean norm.

3. If the system is inconsistent and has a unique least-squares solution, x, the minimal solution is defined to be x.

4. If the system is inconsistent and has set of least-squares solutions, then the minimal solution is the element of this set having the least Euclidean norm.

Theorem 2.6 [15] *The minimal solution of the system (2.4)*
1. is obtained by $x = S^{+}Y$.
2. is a fuzzy vector for an arbitrary fuzzy vector if and only if S^{+} is

non-negative, i.e.

$$(S^+)_{ij}, \quad 1 \leq i \leq 2n, \quad 1 \leq j \leq 2n.$$

Theorem 2.7 [22] *The fuzzy linear system (2.2) is a consistent fuzzy linear system, if and only if* $rank[S] = rank[S|Y]$.

Definition 2.8 [15] *Let* $X(r) = \{\underline{x}_i(r), -\bar{x}_i(r)), 1 \leq i \leq n\}$ *denote a solution of (2.4). The fuzzy number vector* $U = \{\underline{u}_i(r), -\bar{u}_i(r)), 1 \leq i \leq n\}$ *defined by*

$$\begin{bmatrix} \underline{u}_i(r) = \min\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\}, \\ \bar{u}_i(r) = \max\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\} \end{bmatrix}$$

is called a fuzzy solution of (2.4) . If $(\underline{x}_i(r), \bar{x}_i(r)), 1 \leq i \leq n)$ *, are all fuzzy numbers and* $\underline{x}_i(r) = \underline{u}_i(r), \bar{x}_i(r) = \bar{u}_i(r), 1 \leq i \leq n$ *, then* U *is called a strong fuzzy solution. Otherwise,* U *is a weak fuzzy solution.*

3 New Results

The objective of this section is to give the new properties of the index of matrix and Drazin inverse.

Theorem 3.1 *Let* $A \in C^{n\times n}$ *. For any* $n \in N$ *we have* $ind(A^n) \leq ind(A)$ *.*

Proof. From [21] if $\lambda_i; 1 \leq i \leq n$ are eigenvalues of A then $\lambda_i^n; 1 \leq i \leq n$ are eigenvalues of A . Let $\lambda_{1_{(A)}} = 0$ be an eigenvalue of A and be $\sigma(\lambda_{1_{(A)}}) = m$ the multiplicity of the eigenvalue $\lambda_{1_{(A)}}$, then the maximum number of linearly independent eigenvectors associated with $\lambda_{1_{(A)}}$ is

$$\rho_{1_{(A)}} = n - rank(A).$$

Also if $\lambda_{1_{(A^n)}} = 0$ we have $\rho_{1_{(A^n)}} = n - rank(A^n)$. From [21] $rank(A^n) \leq rank(A)$, then

$$\rho_{1_{(A)}}(A) \leq \rho_{1_{(A^n)}}(A^n).$$

$Ind(A)$ is the dimension of largest Jordan block corresponding to the zero eigenvalue of A [26]. Therefore $ind(A^n) \leq ind(A)$.

Theorem 3.2 *For any* $A \in C^{n\times n}$, $ind(A^D) \leq ind(A)$.

Proof. Let $\lambda_{1_{(A)}} = 0$ be an eigenvalue of A and be $\sigma(\lambda_{1_{(A)}}) = m$ the multiplicity of the eigenvalue $\lambda_{1_{(A)}}$, then the maximum number of linearly independent eigenvectors associated with $\lambda_{1_{(A)}}$ is

$$\rho_{1_{(A)}} = n - rank(A).$$

Also if $\lambda_{1_{(A^D)}} = 0$ we have $\rho_{1_{(A^D)}} = n - rank(A^D)$. From [26] $rank(A^D) \leq rank(A)$, then

$$\rho_{1_{(A)}}(A) \leq \rho_{1_{(A^D)}}(A^D).$$

$Ind(A)$ is the dimension of largest Jordan block corresponding to the zero eigenvalue of A [26]. Therefore $ind(A^D) \leq ind(A)$.

Corollary 3.1 *Let* $A \in C^{n\times n}$, $ind(A) = 1$. *By [26]* $rank(A) = rank(A_g)$ *, then* $ind(A) = ind(A_g)$.

Corollary 3.2 *Let* $A \in C^{n\times n}$, $ind(A) = 0$, *then* $ind(A) = ind(A^{-1})$.

Theorem 3.3 *Let* $A \in C^{n\times n}$ *be a singular matrix with index* k, *then* A^{k+1} *is a singular matrix.*

Proof. For any matrix $A \in C^{n\times n}$ by Theorem 3.1, $ind(A^n) < ind(A)$ for $n \in N$. In this case, $ind(A) > 1$ we have $ind(A^{k+1}) = ind(A)$. Thus A^{k+1} is a singular matrix.

4 Solving Singular Fuzzy Linear Systems

In this sections, Ezzati's method is extended and on the singular fuzzy linear systems is performed. Then a method for finding minimal solution of singular fuzzy linear system of equations $A\tilde{x} = \tilde{b}$ when every entry of A be positive number, is given. It is to be note that, the linear system of equations $Ax = b$ where $A = [a_{ij}] \in C^{m\times n}$, $m < n$ is a rectangular crisp matrix, and the right-hand side is a fuzzy vector is called an underdetermined fuzzy linear system of equations.

Generalized Ezzati's method. Consider the following consistent singular fuzzy linear system

$$\begin{cases} a_{11}\tilde{x}_1 + \cdots + a_{1n}\tilde{x}_n = \tilde{y}_1, \\ a_{21}\tilde{x}_1 + \cdots + a_{2n}\tilde{x}_n = \tilde{y}_2, \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}\tilde{x}_1 + \cdots + a_{nn}\tilde{x}_n = \tilde{y}_n. \end{cases} \tag{4.1}$$

For solving consistent system (4.5) we first solve the following system

$$\begin{cases} a_{11}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{1n}(\underline{x}_n + \bar{x}_n) = (\underline{y}_1 + \bar{y}_1), \\ a_{21}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{2n}(\underline{x}_n + \bar{x}_n) = (\underline{y}_2 + \bar{y}_2), \\ \quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}(\underline{x}_1 + \bar{x}_1) + \cdots + a_{nn}(\underline{x}_n + \bar{x}_n) = (\underline{y}_n + \bar{y}_n), \end{cases}$$

and suppose the solution of this system is as

$$d = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} \underline{x}_1 + \bar{x}_1 \\ \vdots \\ \underline{x}_n + \bar{x}_n \end{bmatrix}$$

Let matrices B and C have defined as definition2.6 . According to Ezzati's method we have

$$(B + C)\underline{x}(r) = \underline{y}(r) + Cd, \tag{4.2}$$

and

$$(B + C)\bar{x}(r) = \bar{y}(r) + Cd. \tag{4.3}$$

Theorem 4.1 *If the consistent singular fuzzy linear system (4.5) is replaced by two $n \times n$ crisp linear systems (4.6) and (4.7), then*

$$\underline{x}(r) = (B + C)^D(\underline{y}(r) + Cd), \quad k = ind(B + C),$$

and

$$\bar{x}(r) = (B+C)^D(\bar{y}(r)+Cd), \quad k = ind(B+C),$$

if and only if $(\underline{x}(r)+Cd) \in R((B+C)^k)$ *and* $(\underline{x}(r)+Cd) \in R((B+C)^k)$ *respectively.*

Proof. Same as the proof of theorem2.7.3 in [19].

Therefore by theorem4.1. we can solving singular fuzzy linear systems using generalized Ezzati's method.

Theorem 4.2 *The consistent singular fuzzy linear system of equations*

$$A\tilde{x} = \tilde{b}, \tag{4.4}$$

where $A = [a_{ij}] \in R^{n \times n}$, $a_{ij} > 0$, $ind(A) = k$, $rank(A) = m$ *and* $\tilde{b}$ *is a fuzzy vector, is equivalent to full-rank underdetermined fuzzy linear system of equations.*

Proof. For solving (4.8) we get $2n \times 2n$ linear system

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ a_{n1} & \cdots & a_{nn} & & & \\ & & & a_{11} & \cdots & a_{1n} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \\ -\bar{b}_1 \\ \vdots \\ -\bar{b}_n \end{bmatrix}.$$

Since S is singular matrix by theorem 2.4 we have

$$X_D = S^D Y = \begin{bmatrix} A^D \underline{Y} \\ -A^D \bar{Y} \end{bmatrix}$$

By theorem 2.5, the singular linear system (4.8) is equivalent to the following row-reduced echelon matrix

$$\begin{bmatrix} r_{11} & \cdots & r_{1n} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ r_{m1} & \cdots & r_{mn} & & & \\ & & & r_{11} & \cdots & r_{1n} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & r_{m1} & \cdots & r_{mn} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{bmatrix} = \begin{bmatrix} \underline{c}_1 \\ \vdots \\ \underline{c}_n \\ -\bar{c}_1 \\ \vdots \\ -\bar{c}_n \end{bmatrix} \tag{4.5}$$

The system (4.9) is the full-rank under determined linear system of equations. The systems (4.8) and (4.9) is equivalent. The minimal solution of (4.8) is

$$X_P = \begin{bmatrix} r_{11} & \cdots & r_{1n} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ r_{m1} & \cdots & r_{mn} & & & \\ & & & r_{11} & \cdots & r_{1n} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & r_{m1} & \cdots & r_{mn} \end{bmatrix}^{+} \begin{bmatrix} \underline{c}_1 \\ \vdots \\ \underline{c}_n \\ -\bar{c}_1 \\ \vdots \\ -\bar{c}_n \end{bmatrix} = \begin{bmatrix} R^{+}\underline{C} \\ r - R^{+}\bar{C}. \end{bmatrix}$$

Corollary 4.1 *Equivalent systems of linear equations have exactly the same solutions [28]. Therefore X_p is the minimal solution of the singular fuzzy linear system of equations (4.8).*

Corollary 4.2 *It is clear that the consistent singular linear system of equations has a set solution, and the inconsistent singular linear system of equation has a least squares set solution. Typically, an underdetermined system has an infinite number of solutions [21]. Thus X_D may not be the minimal solution of the underdetermined fuzzy linear system of equations (4.9). However is a solution of it.*

5 Numerical Examples

In this section, we give numerical examples to illustrate previous sections.

Example 5.1 *Consider the following consistent singular fuzzy linear system of equations*

$$\begin{cases} \widetilde{x}_1 + 3\widetilde{x}_2 + \widetilde{x}_3 = (r, 3-r), \\ \widetilde{x}_1 + 3\widetilde{x}_2 + 3\widetilde{x}_3 = (2r, -3r+5), \\ \widetilde{x}_1 + 3\widetilde{x}_2 + \widetilde{x}_3 = (r, 3-r). \end{cases} \tag{5.1}$$

By Theorem 2.3 for computing Drazin inverse of the matrix A we have

$$A = P^{-1}\begin{bmatrix} 6 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P, \quad P = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{3}{7} & -\frac{5}{7} \\ 1 & 0 & -1, \end{bmatrix}$$

thus

$$A^D = P^{-1}\begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P = \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & -\frac{2}{3} \\ -\frac{1}{9} & -\frac{1}{3} & \frac{7}{9} \\ \frac{1}{6} & \frac{1}{2} & -\frac{2}{3}. \end{bmatrix}$$

From Theorem 4.2 we have

$$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ -\bar{x}_1 \\ -\bar{x}_2 \\ -\bar{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{1}{9} & -\frac{1}{3} & \frac{7}{9} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{2} & -\frac{2}{3} \\ 0 & 0 & 0 & -\frac{1}{9} & -\frac{1}{3} & \frac{7}{9} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{2} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} r \\ 2r \\ r \\ -3+r \\ 3r-5 \\ -3+r \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r \\ 0 \\ \frac{1}{2}r \\ -1+r \\ -\frac{1}{3}-\frac{1}{3}r \\ -1+r \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \widetilde{x}_1 = (\frac{1}{2}r, 1-r), \\ \widetilde{x}_2 = (0, \frac{1}{3}r + \frac{1}{3}), \\ \widetilde{x}_3 = (\frac{1}{2}r, 1-r), \end{bmatrix}$$

is the solution of (5.10). Obviously $\widetilde{x}_1$, $\widetilde{x}_2$ *and* $\widetilde{x}_3$ *are not fuzzy numbers. Therefore the corresponding fuzzy solution is a weak fuzzy solution given by*

$$\begin{bmatrix} \widetilde{u}_1 = (0, 1-r), \\ \widetilde{u}_2 = (0, \frac{2}{3}), \\ \widetilde{u}_2 = (0, 1-r). \end{bmatrix}$$

The system $SX = Y$ *and the full-rank underdetermined fuzzy linear system*

$$\begin{bmatrix} 1\ 3\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1\ 3\ 0 \\ 0\ 0\ 0\ 0\ 0\ 1 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ -\bar{x}_1 \\ -\bar{x}_2 \\ -\bar{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r \\ \frac{1}{2}r \\ -2 \\ r-1 \end{bmatrix}, \qquad (5.2)$$

is equivalent by theorem 4.2. Therefore

$$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ -\bar{x}_1 \\ -\bar{x}_2 \\ -\bar{x}_3 \end{bmatrix} = \begin{bmatrix} 1\ 3\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1\ 3\ 0 \\ 0\ 0\ 0\ 0\ 0\ 1 \end{bmatrix}^{+} \begin{bmatrix} \frac{1}{2}r \\ \frac{1}{2}r \\ -2 \\ r-1 \end{bmatrix},$$

is the minimal solution of (5.11). Thus minimal solution of the singular *fuzzy linear system (5.11) is*

$$\begin{bmatrix} \widetilde{x}_1 = (\frac{1}{20}r, \frac{1}{5}), \\ \widetilde{x}_2 = (\frac{3}{20}r, \frac{3}{5}), \\ \widetilde{x}_2 = (\frac{1}{2}r, 1-r). \end{bmatrix}$$

Obviously $\widetilde{x}_3$ is not fuzzy number, and hence we can obtain the weak minimal fuzzy solution as follows

$$\begin{bmatrix} \widetilde{x}_1 = (\frac{1}{20}r, \frac{1}{5}), \\ \widetilde{x}_2 = (\frac{3}{20}r, \frac{3}{5}), \\ \widetilde{x}_2 = (0, 1-r). \end{bmatrix}$$

Example 5.2 *Consider the following consistent singular fuzzy linear system of equations*

$$\begin{cases} 5\widetilde{x}_1 + 10\widetilde{x}_2 &= (12+6r, 27-9r), \\ 10\widetilde{x}_1 + 20\widetilde{x}_2 &= (24+12r, 54-18r). \end{cases} \tag{5.3}$$

By theorem2.3 we have

$$A = P^{-1}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}P, \quad P = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ \frac{4}{5} & -\frac{2}{5} \end{bmatrix},$$

thus

$$A^D = P^{-1}\begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix}P = \begin{bmatrix} \frac{1}{125} & \frac{2}{125} \\ \frac{2}{152} & \frac{4}{125} \end{bmatrix}.$$

Therefore

$$X_D = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{125} & \frac{2}{125} & 0 & 0 \\ \frac{2}{152} & \frac{4}{125} 0 & 0 & \\ 0 & 0 & \frac{1}{125} & \frac{2}{125} \\ 0 & 0 & \frac{2}{152} & \frac{4}{125} \end{bmatrix} \begin{bmatrix} 12+6r \\ 24+12r \\ -27+9r \\ -54+18r \end{bmatrix} = \begin{bmatrix} \frac{12}{25}+\frac{6}{25}r \\ \frac{24}{25}+\frac{12}{25}r \\ -\frac{27}{25}+\frac{9}{25}r \\ -\frac{54}{25}+\frac{18}{25}r \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \widetilde{x}_1 = (\frac{12}{25}+\frac{6}{25}r, \frac{27}{25}-\frac{9}{25}r), \\ \widetilde{x}_2 = (\frac{24}{25}+\frac{12}{25}r, \frac{54}{25}-\frac{18}{25}r), \end{bmatrix}$$

is the solution of (5.12). X_D is a strong fuzzy solutions. The system (5.12) and the full-rank underdetermined fuzzy linear system

$$\begin{bmatrix} 5 & 10 & 0 & 0 \\ 0 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{bmatrix} = \begin{bmatrix} 12+6r \\ 9r-27 \end{bmatrix}, \qquad (5.4)$$

are equivalent. In this case, we show that $X_D = X_P$. By Theorem 4.2

$$X_P = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 & 0 \\ 0 & 0 & 5 & 10 \end{bmatrix}^+ \begin{bmatrix} 12+6r \\ 9r-27 \end{bmatrix} = \begin{bmatrix} \frac{12}{25}+\frac{6}{25}r \\ \frac{24}{25}+\frac{12}{25}r \\ -\frac{27}{25}+\frac{9}{25}r \\ -\frac{54}{25}+\frac{9}{18}r \end{bmatrix},$$

is the minimal solution of (5.12). Thus

$$\begin{bmatrix} \widetilde{x}_1 = (\frac{12}{25}+\frac{6}{25}r, \frac{27}{25}-\frac{9}{25}r), \\ \widetilde{x}_2 = (\frac{24}{25}+\frac{12}{25}r, \frac{54}{25}-\frac{9}{18}r), \end{bmatrix}$$

is the minimal solution of the singular fuzzy linear system (5.12).

Example 5.3 *Consider the following consistent singular fuzzy linear system of equations*

$$\begin{cases} \widetilde{x}_1 - \widetilde{x}_2 &= (2+r, 3), \\ 3\widetilde{x}_1 - 3\widetilde{x}_2 &= (6+3r, 9). \end{cases} \tag{5.5}$$

By generalized Ezzati's method we can get the following system

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \underline{x}_1 + \bar{x}_1 \\ \underline{x}_2 + \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 5+r \\ 15+3r \end{bmatrix}.$$

By theorem2.3 we have

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} = P^{-1} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} P, \quad P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Therefore we can get thus

$$\begin{bmatrix} \underline{x}_1 + \bar{x}_1 \\ \underline{x}_2 + \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}^D \begin{bmatrix} 5+r \\ 15+3r \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} - \frac{1}{2}r \\ -\frac{15}{2} - \frac{3}{2}r \end{bmatrix}.$$

By theorem2.3 we have

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = P^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} P, \quad P = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}.$$

We get $ind(B+C) = 1$ *then*

$$\underline{x}(r) = \begin{bmatrix} \frac{1}{16} & \frac{1}{16} \\ \frac{3}{16} & \frac{3}{16} \end{bmatrix} \begin{bmatrix} \underline{y} + Cd \end{bmatrix} = \begin{bmatrix} -\frac{11}{8} - \frac{1}{8}r \\ -\frac{33}{8} - \frac{3}{8}r \end{bmatrix},$$

and

$$\bar{x}(r) = \begin{bmatrix} \frac{1}{16} & \frac{1}{16} \\ \frac{3}{16} & \frac{3}{16} \end{bmatrix} \begin{bmatrix} \bar{y} + Cd \end{bmatrix} = \begin{bmatrix} -\frac{9}{8} - \frac{3}{8}r \\ -\frac{27}{8} - \frac{9}{8}r \end{bmatrix}.$$

is a solution of (5.14).

6 Conclusions and Suggestions

In this paper, solving singular fuzzy linear system of equations $A\widetilde{x} = \widetilde{b}$ is investigated. Generalized Ezzati's method for solving such system is given. A method for finding minimal solution of singular fuzzy linear system of equations $A\widetilde{x} = \widetilde{b}$ while A be a crisp matrix with positive elements is given. Solving singular fuzzy linear system of equations by iterative methods is suggested.

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