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Interpolation of the tabular functions with fuzzy input and fuzzy output

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Abstract

In this paper, first a design is proposed for representing fuzzy polynomials with input fuzzy and output fuzzy. Then, we sketch a constructive proof for existence of such polynomial which can be fuzzy interpolation polynomial in a set given of discrete points rather than a fuzzy function. Finally, to illustrate some numerical examples are solved.

Key words: Fuzzy polynomials, Fuzzy numbers, Approximation polynomial, Fuzzy interpolation.

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1 Introduction

Finding the approximation or interpolation polynomial of functions in a given space is one of the interesting, important and attractive problems in applied mathematics. The interpolation with fuzzy date first was presented by Zadeh [1].

Up to now, many researchers have worked on this problem. A fuzzy Lagrange interpolation Theorem is given in [2]. The some properties of fuzzy Lagrange and cubic spline interpolation are given in [3]. Abbasbandy and et.al [4,5], had interpolation problem by natural splines and complete splines. Also, they had obtained some approximation methods of fuzzy functions on a set given of points $(x_i, \widetilde{f}_i)$, which $\widetilde{f}_i$ is a triangular fuzzy number for $i = 0, 1, \cdots, n$, by applying a defuzzification function and translating problem to linear programming problems, (see [6] and [7]). A approximation method of fuzzy functions with fuzzy input and fuzzy output by using fuzzy splines series is given in [8].

In this paper, we first, will define a pattern for fuzzy polynomials of degree at most n with fuzzy variable. We sketch a constructive proof to show that a such as polynomial exists. Then, we approximating a fuzzy function with fuzzy input and fuzzy output on a set given of discrete points $(\widetilde{x_i}, \widetilde{f}_i)$, $i = 0, 1, \cdots, n$ and a set of them α-cuts.

This paper is organized as follows:

In section 2, we introduce the basic concepts of fuzzy numbers. Then, we prepare a similitude for representing of fuzzy numbers based on themes parametric representation. Also, a pattern for representing of fuzzy polynomials is introduced. The fuzzy approximation problem is introduced in section 3, constructing fuzzy polynomial, proving its existence and approximating of fuzzy interpolation polynomial in a set given of fuzzy data are given in details. Finally, some examples are given in section 4, and conclusion in section 5.

2 Preliminaries

The parametric form of a fuzzy number is as $\widetilde{a} = (\underline{a}(\alpha), \overline{a}(\alpha))$, for each $\alpha \in [0, 1]$, where the following requirements;
1. $\underline{a}(\alpha)$ is monotonically increasing left continuous function.
2. $\overline{a}(\alpha)$ is monotonically decreasing left continuous function.
3. $\underline{a}(\alpha) \leq \overline{a}(\alpha)$, $0 \leq \alpha \leq 1$.

Let $\mathbb{R}_F$ be the set of all fuzzy numbers. We call that the number $\widetilde{a} = (\underline{a}(\alpha), \overline{a}(\alpha)) \in \mathbb{R}_F$ is positive if $\underline{a}(0) \geq 0$ and it is negative if $\overline{a}(0) \leq 0$. If $\widetilde{b} = (\underline{b}(\alpha), \overline{b}(\alpha)) \in \mathbb{R}_F$ then, some results of applying fuzzy arithmetic on two fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ are as follows:

$$\lambda\widetilde{a} = \begin{cases} (\lambda\underline{a}(\alpha), \lambda\overline{a}(\alpha));\ \lambda \geq 0, \\ (\lambda\overline{a}(\alpha), \lambda\underline{a}(\alpha))\ \ \lambda < 0, \end{cases} \tag{2.1}$$

$$\widetilde{a} + \widetilde{b} = (\underline{a}(\alpha) + \underline{b}(\alpha), \overline{a}(\alpha) + \overline{b}(\alpha)), \tag{2.2}$$

$$\widetilde{a} - \widetilde{b} = (\underline{a}(\alpha) - \overline{b}(\alpha), \overline{a}(\alpha) - \underline{b}(\alpha)),$$

$$\widetilde{a} \cdot \widetilde{b} = \begin{cases} \Big(\underline{a}(\alpha)\underline{b}(\alpha), \overline{a}(\alpha)\overline{b}(\alpha)\Big)\ ;\ \widetilde{a}, \widetilde{b} > 0, \\ \Big(\overline{a}(\alpha)\overline{b}(\alpha), \underline{a}(\alpha)\underline{b}(\alpha)\Big)\ ;\ \widetilde{a}, \widetilde{b} < 0, \\ \Big(\overline{a}(\alpha)\underline{b}(\alpha), \underline{a}(\alpha)\overline{b}(\alpha)\Big)\ ;\ \widetilde{a} > 0,\ \widetilde{b} < 0. \end{cases} \tag{2.3}$$

Please notice that in this paper we not need to dividing two fuzzy numbers.

For each $0 < \alpha \leq 1$, the set α-cut of $\widetilde{a} \in \mathbb{R}_F$, is the interval $[\widetilde{a}]^{\alpha} = [\underline{a}(\alpha), \overline{a}(\alpha)]$. If $\widetilde{a} = (\underline{a}(\alpha), \overline{a}(\alpha))$ be a fuzzy number such that $\underline{a}(1) = \overline{a}(1)$, then its called a central fuzzy number. Otherwise, its called a interval fuzzy number. In this study, we work to central fuzzy numbers and denote by $\mathbb{R}_F^C$ the set of all central fuzzy numbers. A crisp number x, can be simply represented by $\underline{x}(\alpha) = \overline{x}(\alpha) = a,\ \ 0 \leq \alpha \leq 1$

A special class of fuzzy numbers in $\mathbb{R}_F^C$, is the class of triangular fuzzy numbers. Given $\widetilde{a} = (a_c, a_\ell, a_r)$ determined by $a_c \in \mathbb{R}$ and a_ℓ, $a_r \in [0, +\infty)$. Let us denote $T\mathbb{R}_F$ as the set of all triangular fuzzy numbers. $\widetilde{a} \in$

$T\mathbb{R}_F$ is called symmetric if $a_\ell = a_r$. If $\widetilde{a} \in T\mathbb{R}_F$ then $\underline{a}(\alpha) = a_c - a_\ell(1-\alpha)$ and $\overline{a}(\alpha) = a_c + a_r(1-\alpha)$, for each $\alpha \in [0,1]$.

Now, suppose that $\widetilde{a} \in \mathbb{R}_F^C$, similarly to representation of a triangular fuzzy number, we prefer to employ the following form for showing $\widetilde{a}$:

$$\widetilde{a} = (a_c, a_\ell(\alpha), a_r(\alpha)), \quad \forall\, \alpha \in [0,1]. \tag{2.4}$$

Where $a_c = \underline{a}(1) = \overline{a}(1)$ is called Core or Pick of $\widetilde{a}$ and

$$a_\ell(\alpha) = a_c - \underline{a}(\alpha), \quad a_r(\alpha) = \overline{a}(\alpha) - a_c. \tag{2.5}$$

It is clear that $a_\ell(\alpha)$ and $a_r(\alpha)$ are positive and decreasing functions which are left and right spreads of fuzzy number $\widetilde{a}$ for each $\alpha \in [0,1]$. The representation (2.4) of fuzzy numbers is useful in practical because we can observe effect of arithmetic operators on core points and left and right spreads of fuzzy numbers. Moreover, if $\widetilde{a} = (a_c, a_\ell(\alpha), a_r(\alpha))$ is a triangular fuzzy number then we have

$$a_\ell(\alpha) = a_\ell(1-\alpha), \quad a_r(\alpha) = a_r(1-\alpha)$$

it follows that $\widetilde{a}$ can be represented independent of α i.e. $\widetilde{a} = (a_c, a_\ell, a_r)$ where $a_\ell = a_\ell(0)$ and $a_r = a_r(0)$.

Theorem 2.1 *If $\widetilde{a} = (a_c, a_\ell(\alpha), a_r(\alpha))$ and $\widetilde{b} = (b_c, b_\ell(\alpha), b_r(\alpha)) \in \mathbb{R}_F^C$, then*

$$\lambda\widetilde{a} = \begin{cases} (\lambda a_c, \lambda a_\ell(\alpha), \lambda a_r(\alpha)); & \lambda \geq 0, \\ (\lambda a_c, -\lambda a_r(\alpha), -\lambda a_\ell(\alpha)); & \lambda < 0, \end{cases} \tag{2.6}$$

$$\widetilde{a} + \widetilde{b} = (a_c + b_c, a_\ell(\alpha) + b_\ell(\alpha), a_r(\alpha) + b_\ell(\alpha)), \tag{2.7}$$

$$\widetilde{a} - \widetilde{b} = (a_c - b_c, a_\ell(\alpha) + b_r(\alpha), a_r(\alpha) + b_\ell(\alpha)),$$

$$
\widetilde{a}\cdot\widetilde{b}=\begin{cases}\Big(a_cb_c, a_cb_\ell(\alpha)+b_ca_\ell(\alpha)-a_\ell(\alpha)b_\ell(\alpha),\\ a_cb_r(\alpha)+b_ca_r(\alpha)+a_r(\alpha)b_r(\alpha)\Big); & \widetilde{a},\widetilde{b}>0,\\ \Big(a_cb_c, -a_cb_r(\alpha)-b_ca_r(\alpha)-a_r(\alpha)b_r(\alpha),\\ -a_cb_\ell(\alpha)-b_ca_\ell(\alpha)+a_\ell(\alpha)b_\ell(\alpha)\Big); & \widetilde{a},\widetilde{b}<0,\\ \Big(a_cb_c, a_cb_\ell(\alpha)-b_ca_r(\alpha)+a_r(\alpha)b_\ell(\alpha),\\ a_cb_r(\alpha)-b_ca_\ell(\alpha)-a_\ell(\alpha)b_r(\alpha)\Big); & \widetilde{a}>0,\widetilde{b}<0.\end{cases} \tag{2.8}
$$

Proof. By using of Definitions (2.1),(2.2) and (2.3) and also relationships (2.5) the proof is simple.

Remark 2.2 *The arithmetic operators in Theorem 2.1 are similar to arithmetic operators on L-R fuzzy numbers (see [9]). But the fundamental idea of the L-R representation of fuzzy numbers is to split the membership function of a fuzzy number into two curves, left and right of the model value. In L-R computations the results are as approximation, usually. For example, multiplication two L-R fuzzy numbers from approximated as a L-R fuzzy number.*

In what follows that, we proposed a design for general representation of the fuzzy polynomials with fuzzy input and fuzzy output.

Definition 2.3 *Suppose that D be a non empty subset of $\mathbb{R}_F$. A fuzzy polynomial on D from degree at most n, is the function $\widetilde{P}:D\to\mathbb{R}_F^C$ which*

$$
\widetilde{P}(\widetilde{x})=\Big(P_c(x_c), P_\ell(x_c,\alpha;x_\ell(\alpha),x_r(\alpha)), P_r(x_c,\alpha;x_\ell(\alpha),x_r(\alpha))\Big) \tag{2.9}
$$

where $P_c(x)\in\prod_n$ and $\prod_n$ is the set all crisp polynomials from degree at most n. Two functions P_ℓ and P_r are both nonnegative polynomials and decreasing with respect to α, that are dependent to variables x_c,α and the functions $x_\ell(\alpha),x_r(\alpha)$, in general case. We denote by $\widetilde{\prod}_n$ the set all such fuzzy polynomials of degree at most n. Also, if $D_{\widetilde{p}}$ be the greatest as the set D, then it called domain of $\widetilde{p}$.

Remark 2.4 *The representation (2.9) of fuzzy polynomials is a extension of representation of fuzzy polynomials with the real variable. Indeed, if* $\widetilde{a}_i = (a_{ic}, a_{i\ell}(\alpha), a_{ir}(\alpha)) \in \mathbb{R}_F^C,\quad i = 0, 1, \cdots, n$ *are the coefficient of polynomial then, for any real number* x*, we have*

$$
\begin{aligned}
\widetilde{p}(x) &= \sum_{i=0}^{n} \widetilde{a}_i x^i \\
&= \Big(\sum_{i=0}^{n} a_{ic} x^i, \sum_{x^i \geq 0} a_{i\ell}(\alpha) x^i - \sum_{x^i \leq 0} a_{ir}(\alpha) x^i, \\
&\quad \sum_{x^i \geq 0} a_{ir}(\alpha) x^i - \sum_{x^i \leq 0} a_{i\ell}(\alpha) x^i \Big) \\
&= (p_c(x), p_\ell(x, \alpha), p_r(x, \alpha)).
\end{aligned}
$$

It is clear that the functions $p_\ell(x, \alpha)$ *and* $p_r(x, \alpha)$ *are nonnegative and decreasing with respect to* α *and* $p_c(x) \in \prod_n$*. However, based on Definition 2.3, it is possible that, a fuzzy polynomial not be defined on* $\mathbb{R}_F^C$*.*

Example 2.5 *The following functions*

$$\widetilde{p}(\widetilde{x}) = \Big(x_c^2, 2x_c x_\ell(\alpha) - x_\ell^2(\alpha), 2x_c x_r(\alpha) + x_r^2(\alpha) \Big),$$

$$\widetilde{q}(\widetilde{x}) = \Big(x_c^2, -2x_c x_r(\alpha) - x_r^2(\alpha), -2x_c x_\ell(\alpha) + x_\ell^2(\alpha) \Big),$$

are fuzzy polynomials that are defined, on positive and negative fuzzy numbers, respectively. We have

$$
\begin{aligned}
\widetilde{p}(\widetilde{x}) &= \Big(x_c^2, x_c^2 - (x_c - x_\ell(\alpha))^2, (x_r(\alpha) + x_c)^2 - x_c^2 \Big) \\
&= \Big(x_c^2, x_c^2 - \underline{x}^2(\alpha), \overline{x}^2(\alpha) - x_c^2 \Big) \\
&= \Big[\underline{x}^2(\alpha), \overline{x}^2(\alpha) \Big] = \widetilde{x}^2,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}(\widetilde{x}) &= \Big(x_c^2, x_c^2 - (x_c + x_r(\alpha))^2, (x_c - x_\ell(\alpha))^2 - x_c^2 \Big) \\
&= \Big(x_c^2, x_c^2 - \overline{x}^2(\alpha), \underline{x}^2(\alpha) - x_c^2 \Big) \\
&= \Big[\overline{x}^2(\alpha), \underline{x}^2(\alpha) \Big] = \widetilde{x}^2.
\end{aligned}
$$

Definition 2.6 *We say that two numbers* $\widetilde{a}$ *and* $\widetilde{b} \in \mathbb{R}_F^C$, *are distinct everywhere* $a_c \neq b_c$.

3 Approximation of the fuzzy interpolation polynomial

Let $\widetilde{x} = \{\widetilde{x}_0, \widetilde{x}_1, \cdots, \widetilde{x}_n\}$ be a set of n points in $\mathbb{R}_F^C$, and the values of a function $f : \mathbb{R}_F^C \to \mathbb{R}_F^C$ at these points are $\widetilde{f}(\widetilde{x}_i) = (f_{ic}, f_{i\ell}(\alpha), f_{ir}(\alpha))$, that is $(\widetilde{x}_i, \widetilde{f}_i) \in \mathbb{R}_F^C \times \mathbb{R}_F^C$ for $i = 0, 1, \cdots, n$ are given. Let $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_m \leq 1$ are given. The problem is finding a fuzzy polynomial belong to $\widetilde{\prod}_n$, such that

$$[\widetilde{P}(\widetilde{x}_i)]^{\alpha_k} = [\widetilde{f}(\widetilde{x}_i)]^{\alpha_k}, \quad i = 0, 1, \cdots, n, \quad k = 0, 1, \cdots, m. \tag{3.1}$$

Please remark that, in general case, the value m is depended to figure of $\widetilde{f}_i, \quad i = 0, 1, \cdots, n$. For example if they are triangular fuzzy numbers then it is sufficient that we choose $m = 1$.

Example 3.1 *Assume that* $n = 0$ *and consider the point* $(\widetilde{x}_0, \widetilde{f}_0)$ *with* $\widetilde{x}_0 = (x_c, \lambda, \lambda)$ *and* $\widetilde{f}_0 = (x_c^k, \lambda^k, \lambda^k)$ *as symmetric triangular fuzzy numbers, where* $\lambda \geq 0$ *and* $k \in \{1, 2, \cdots\}$. *The both functions*

$$\widetilde{p}(\widetilde{x}) = \left(x_c^k, x_\ell(\alpha)^k, x_r(\alpha)^k\right),$$

and

$$\widetilde{q}(\widetilde{x}) = \left(x_c^k, x_r(\alpha)^k, x_\ell(\alpha)^k\right),$$

are fuzzy polynomials and defined on $\mathbb{R}_F^C$. *We observe that*

$$\widetilde{p}(\widetilde{x}_0) = \widetilde{q}(\widetilde{x}_0) = (x_c^k, \lambda^k, \lambda^k) = \widetilde{f}_0,$$

it follows that the fuzzy polynomial in form (2.9), such that it satisfies equalities (3.1) is not uniquely.

Theorem 3.2 *Suppose that* $(\widetilde{x}_i, \widetilde{f}_i), \quad i = 0, 1, \cdots, n$ *are a set given of fuzzy data which* $(\widetilde{x}_i, \widetilde{f}_i) \in \mathbb{R}_F^C \times \mathbb{R}_F^C$ *and the fuzzy numbers* $\widetilde{x}_i$ *are disjunct for* $i = 0, 1, \cdots, n$. *Then for the integer number* $m \geq 0$ *a fuzzy polynomial of degree at most* n, *in form (2.9) exists, such that satisfies in equalities (3.1).*

Proof. Let $\widetilde{x}_i = (x_{ic}, x_{i\ell}(\alpha), x_{ir}(\alpha))$ and $\widetilde{f}_i = (f_{ic}, f_{i\ell}(\alpha), f_{ir}(\alpha)) \in \mathbb{R}_F^C$ for $i = 0, 1, \cdots, n$. Consider a set given of crisp points (x_{ic}, f_{ic}), $i = 0, 1, \cdots, n$. We know that there exists crisp polynomial P_c such that $P_c(x_{ic}) = f_{ic}$ and P_c is uniquely of degree at most n. We construct two functions P_ℓ and P_r, such that for $i = 0, 1, \cdots, n$ and $k = 0, 1, \cdots, m$, we have

$$P_\ell(x_{ci}, \alpha_k; x_{\ell i}(\alpha_k), x_{ri}(\alpha_k)) = f_{\ell i}(\alpha_k),$$

and

$$P_r(x_{ci}, \alpha_k; x_{\ell i}(\alpha_k), x_{ri}(\alpha_k)) = f_{ri}(\alpha_k).$$

For this end, let us consider $\alpha_k \in A$ for $k \in \{0, 1, ..., m\}$, which is fixed. For $i = 0, 1, \cdots, n$ we define a set of indexes as the following from

$$J_{i\ell}(k) = \{J_{0\ell}(k), J_{1\ell}(k), \cdots, J_{t\ell}(k)\},$$

such that:

$$x_{i\ell}(\alpha_k) = x_{J_{P\ell}(k)\ell}(\alpha_k), \quad P = 0, 1, \cdots, t.$$

For $i \in \{0, 1, \cdots, n\}$ we design $n+1$ functions $\varphi_{i\ell}^k$ as the following form

$$\varphi_{i\ell}^k(x_c, x_\ell(\alpha_k)) = \prod_{\substack{j \in J_{i\ell}(k) \\ j \neq i}} \left(\frac{x_c - x_{jc}}{x_{ic} - x_{jc}}\right) \prod_{\substack{j \notin J_{i\ell}(k) \\ j \neq i}} \left(\frac{x_\ell(\alpha_k) - x_{j\ell}(\alpha_k)}{x_{i\ell}(\alpha_k) - x_{j\ell}(\alpha_k)}\right).$$

It is clear that, for every $k \in \{0, 1, \cdots, m\}$ the function $\varphi_{i\ell}^{(k)}$ is a polynomial from two variables x_c and $x_\ell(\alpha_k)$ and summation of its degrees with respect to these variables is at most n. Also, we have

$$\varphi_{i\ell}^{(k)}(x_{jc}, x_{j\ell}(\alpha_k)) = \delta_{ij} = \begin{cases} 1; \ i = j, \\ 0; \ i \neq j, \end{cases}$$

where $i, j \in \{0, 1, \cdots, n\}$. Now, we construct the function P_ℓ as follows

$$P_\ell(x_c, \alpha; x_\ell(\alpha)) = \sum_{k=0}^{m} \sum_{i=0}^{n} L_k(\alpha) \varphi_{i\ell}^k(x_c, x_\ell(\alpha_k)) f_{i\ell}(\alpha_k). \qquad (3.2)$$

Where $L_k(\alpha) = \prod_{\substack{j=0 \\ j \neq k}}^{m} \left(\frac{\alpha - \alpha_j}{\alpha_k - \alpha_j}\right)$ for $k = 0, 1, \cdots, m$ are well-known Lagrang Polynomials.

Structuring the function P_r is similar to the function P_ℓ. Indeed, first we design the following function;

$$\varphi_{i\ell}^{(k)}(x_c, x_r(\alpha_k)) = \prod_{\substack{j\in J_{ir}(k)\\ j\neq i}} \left(\frac{x_c - x_{jc}}{x_{ic} - x_{jc}}\right) \prod_{\substack{j\notin J_{ir}(k)\\ j\neq i}} \left(\frac{x_r(\alpha_k) - x_{jr}(\alpha_k)}{x_{ir}(\alpha_k) - x_{jr}(\alpha_k)}\right),$$

where $J_{ir}(k) = \{J_{0r}(k), J_{1r}(k), \cdots, J_{sr}(k)\}$ such that $x_{ir}(\alpha_k) = x_{J_{ir}}(\alpha_k)$, $i = 0, 1, \cdots, n$. Next, we construct the function P_r by

$$P_r(x_c, x_r(\alpha); \alpha) = \sum_{k=0}^{m} \sum_{i=0}^{n} L_k(\alpha)\varphi_{ir}^{k}(x_c, x_r(\alpha_k)) f_{ir}(\alpha_k). \tag{3.3}$$

In general case nonnegativity and monotony of the functions P_ℓ and P_r are depended to the point $\widetilde{x}$ and the values α_i, $i = 0, 1, \cdots, m$. For sure nonnegativity these functions we define the following representation for required polynomial

$$\widetilde{P}(\widetilde{x}) = \Big(P_c(x_c), |P_\ell(x_c, \alpha; x_\ell(\alpha))|, |P_r(x_c, \alpha; x_r(\alpha))|\Big). \tag{3.4}$$

It is easy to check that $\widetilde{P}$ is satisfied in equalities (3.1). Also, it is clear that if x be a crisp number then $\widetilde{P}(x) \in \mathbb{R}_F^C$. Consequently, $\widetilde{P} \in \widetilde{\prod}_n$ and the Proof is completed.

Remark 3.3 *If we have $(\widetilde{x}_i, \widetilde{f}_i) \in T\mathbb{R}_F^C \times T\mathbb{R}_F^C$, $i = 0, 1, \cdots, n$ then, we will deal to a system with fuzzy linear input fuzzy linear output. So, the functions P_ℓ and P_ℓ, from (3.2) and (3.3), are independent of parameter α, respectively. Therefore, we can rewrite the formula (3.2) and (3.3), as simplified follows*

$$P_\ell(x_c, x_\ell) = \sum_{i=0}^{n} \varphi_{i\ell}(x_c, x_\ell) f_{i\ell}, \tag{3.5}$$

$$P_r(x_c, x_r) = \sum_{i=0}^{n} \varphi_{ir}(x_c, x_r) f_{ir}. \tag{3.6}$$

In this situation, we will have

$$\widetilde{P}(\widetilde{x}) = \Big(P_c(x_c), |P_\ell(x_c, x_\ell)|, |P_r(x_c, x_r)|\Big),$$

that is fuzzy interpolation Polynomial of the function $\widetilde{f}$ *at points* $\widetilde{x}_i = (x_{ic}, x_{i\ell}, x_{ir}) \in T\mathbb{R}_F$ *for* $i = 0, 1, \cdots, n$.

Remark 3.4 *If we have* $(x_i, \widetilde{f}_i) \in \mathbb{R} \times \mathbb{R}_F^C, i = 0, 1, \cdots, n$ *then, we will deal to a system with fuzzy output. In this case we will have* $J_{i\ell}^k = J_{ir}^k = \emptyset$ *for* $i = 0, 1, \cdots, n$ *and* $k = 0, 1, \cdots, m$*. Consequently,* $\widetilde{p}(x)$ *from 3.4 will be a fuzzy interpolation polynomial in given points. Furthermore, if we have* $(x_i, f_i) \in \mathbb{R} \times \mathbb{R}, i = 0, 1, \cdots, n$ *then the polynomial 3.4 transform to well-known Lagrang interpolation polynomial.*

4 Numerical examples

Example 4.1 *Consider three fuzzy data as triangular fuzzy numbers*

$\widetilde{x}_i$	$(1, 1, 1)$	$(3, 1, 2)$	$(4, 2, 1)$
$\widetilde{f}_i$	$(3, 2, 1)$	$(7, 1, 2)$	$(8, 2, 2)$

The Polynomial $P_c(x_c) = 3 + 2(x_c - 1) - \frac{1}{3}(x_c - 1)(x_c - 3)$ *is interpolation polynomial at points* $(1, 3)$*,* $(3, 7)$ *and* $(4, 8)$*. For the sake of applying formula (3.5) and (3.6), we first obtain*

$$J_{0\ell} = J_{0r} = J_{1\ell} = J_{2r} = \{1\}, \quad J_{1r} = J_{2\ell} = \phi,$$

that follows that

$$\varphi_{0\ell}(x_c, x_\ell) = \frac{1}{2}(x_c - 3)(x_\ell - 2),$$
$$\varphi_{1\ell}(x_c, x_\ell) = -\frac{1}{2}(x_c - 1)(x_\ell - 2),$$
$$\varphi_{2\ell}(x_c, x_\ell) = (x_\ell - 1)^2,$$
$$\varphi_{0r}(x_c, x_r) = \frac{1}{3}(x_c - 4)(x_r - 2),$$
$$\varphi_{1r}(x_c, x_r) = (x_r - 1)^2,$$
$$\varphi_{2r}(x_c, x_r) = -\frac{1}{3}(x_c - 1)(x_r - 2).$$

Consequently, we obtain

$$P_\ell(x_c, x_\ell) = (x_c - 3)(x_\ell - 2) - \frac{1}{2}(x_c - 1)(x_\ell - 2) + 2(x_\ell - 1)^2,$$

and

$$P_r(x_c, x_r) = \frac{1}{3}(x_c - 4)(x_r - 2) - \frac{2}{3}(x_c - 1)(x_r - 2) + 2(x_r - 1)^2.$$

Example 4.2 *$n = 2$, $m = 2$*

$\widetilde{x}_i$	$(1, 2 - 2\sqrt{\alpha}, 3 - \sqrt{\alpha})$	$(1.5, 2 - 2\sqrt{\alpha}, 5 - 2\sqrt{\alpha})$	$(2, 1 - \sqrt{\alpha}, 5 - \sqrt{\alpha})$
$\widetilde{f}_i$	$(2, 2 - 2\alpha, 6 - 2\alpha)$	$(3, 1 - \alpha, 8 - 2\alpha)$	$(4, 2 - 2\alpha, 9 - \alpha)$

It is clear that $P_c(x_c) = 2x_c$. Let us, to choose $\alpha_0 = 0$ and $\alpha_1 = \frac{1}{2}$. In this case, we get

$$J_{0\ell}(0) = J_{0\ell}(1) = J_{2r}(0) = \{1\},$$
$$J_{1\ell}(0) = J_{1\ell}(1) = \{0\},$$
$$J_{1r}(0) = \{2\},$$
$$J_{2\ell}(0) = J_{2\ell}(1) = J_{0r}(0) = J_{0r}(1) = J_{1r}(1) = J_{2r}(1) = \phi.$$

Now, we go to obtain the functions $\varphi_{i\ell}^{(k)}$ and $\varphi_{ir}^{(k)}$ for $i = 0, 1, 2$ and $k = 0, 1$. By doing computations we will find

$$
\begin{aligned}
&\varphi_{0\ell}^{(0)}(x_c, x_\ell(0)) = -2(x_c - 1.5)(x_\ell(0) - 1),\\
&\varphi_{0\ell}^{(1)}(x_c, x_\ell(\frac{1}{2})) = \frac{-4}{2-\sqrt{2}}(x_c - 1.5)(x_\ell(\frac{1}{2}) + \frac{\sqrt{2}}{2} - 1),\\
&\varphi_{1\ell}^{(0)}(x_c, x_\ell(0)) = 2(x_c - 1)(x_\ell(0) - 1),\\
&\varphi_{1\ell}^{(1)}(x_c, x_\ell(\frac{1}{2})) = \frac{4}{2-\sqrt{2}}(x_c - 1)(x_\ell(\frac{1}{2}) + \frac{\sqrt{2}}{2} - 1),\\
&\varphi_{2\ell}^{(0)}(x_c, x_\ell(0)) = (x_\ell(0) - 2)^2,\\
&\varphi_{2\ell}^{(1)}(x_c, x_\ell(\frac{1}{2})) = \frac{4}{(2-\sqrt{2})^2}(x_\ell(\frac{1}{2}) + \sqrt{2} - 2)^2,\\
&\varphi_{0r}^{(0)}(x_c, x_r(0)) = \frac{1}{4}(x_r(0) - 5)^2,\\
&\varphi_{0r}^{(1)}(x_c, x_r(\frac{1}{2})) = \frac{1}{4-\sqrt{2}}(x_r(\frac{1}{2}) - 5 + \sqrt{2})(x_r(\frac{1}{2}) - 5 + \frac{\sqrt{2}}{2}),\\
&\varphi_{1r}^{(0)}(x_c, x_r(0)) = -(x_c - 2)(x_r(0) - 3),\\
&\varphi_{1r}^{(1)}(x_c, x_r(\frac{1}{2})) = \frac{2}{1-2\sqrt{2}}(x_r(\frac{1}{2}) - 3 + \frac{\sqrt{2}}{2})(x_r(\frac{1}{2}) - 5 + \frac{\sqrt{2}}{2}),\\
&\varphi_{2r}^{(0)}(x_c, x_r(0)) = (x_c - 1.5)(x_r(0) - 3),\\
&\varphi_{2r}^{(1)}(x_c, x_r(\frac{1}{2})) = \frac{\sqrt{2}}{2}(x_r(\frac{1}{2}) - 3 + \frac{\sqrt{2}}{2})(x_r(\frac{1}{2}) - 5 + \sqrt{2}).
\end{aligned}
$$

Therefore, from (3.2) and (3.3) we obtain, respectively

$$
\begin{aligned}
P_\ell(x_c, \alpha; x_\ell(\alpha)) = {}& (1 - 2\alpha)\Big(- 4(x_c - 1.5)(x_\ell(0) - 1)\\
&+ 2(x_c - 1)(x_\ell(0) - 1) + 2(x_\ell(0) - 2)^2\Big)\\
&+ 2\alpha\Big(\frac{-4}{2-\sqrt{2}}(x_c - 1.5)(x_\ell(\frac{1}{2}) + \frac{\sqrt{2}}{2} - 1)\\
&+ \frac{2}{2-\sqrt{2}}(x_c - 1)(x_\ell(\frac{1}{2}) + \frac{\sqrt{2}}{2} - 1)\\
&+ \frac{4}{(2-\sqrt{2})^2}(x_\ell(\frac{1}{2}) + \sqrt{2} - 2)^2\Big),
\end{aligned}
$$

and

$$\begin{aligned}P_r(x_c,\alpha;x_r(\alpha)) = &(1-2\alpha)\Big(\frac{3}{2}(x_r(0)-5)^2-8(x_c-2)(x_r(0)-3)\\&+9(x_c-1.5)(x_r(0)-3)\Big)\\&+2\alpha\Big(\frac{5}{4-\sqrt{2}}(x_r(\frac{1}{2})-5+\sqrt{2})(x_r(\frac{1}{2})-5+\frac{\sqrt{2}}{2})\\&+\frac{14}{1-2\sqrt{2}}(x_r(\frac{1}{2})-3+\frac{\sqrt{2}}{2})(x_r(\frac{1}{2})-5+\frac{\sqrt{2}}{2})\\&+\frac{17\sqrt{2}}{4}(x_r(\frac{1}{2})-3+\frac{\sqrt{2}}{2})(x_r(\frac{1}{2})-5+\sqrt{2})\Big).\end{aligned}$$

It is easy to check that

$$P_\ell(x_{ic},\alpha;x_{i\ell}(\alpha)) = f_{i\ell}(\alpha), \quad i=0,1,2$$

and

$$P_r(x_{ic},\alpha;x_{ir}(\alpha)) = f_{ir}(\alpha), \quad i=0,1,2.$$

Consequently, $\widetilde{p} = (p_c, |p_\ell|, |p_r|)$ *is an interpolation fuzzy polynomial of the function* $\widetilde{f}$ *in the given points.*

5 Conclusion

In this paper, we have proposed a method for fuzzy interpolation polynomial approximation from a fuzzy tabular function with fuzzy input fuzzy output. We have a proof of existence of a such as polynomial. Our method has a straightforward computational process to obtaining the fuzzy polynomial and also, against previous methods, it is not necessary to be solve one linear or nonlinear programming problem.

References

[1] L. A. Zadeh, Fuzzy sets, Inform. Control (8), (1965), 338-353.

[2] R. Lowen, A Fuzzy Lagrange interpolation theorem, Fuzzy Sets Syst. 34 (1990) 33-34.

[3] O. Kaleva, Interpolation of fuzzy data, Fuzzy Sets and Syst. 61 (1994) 63-70.

[4] S. Abbasbandy, E. Rabolian, Interpolation of fuzzy data by natural splines, J. Appl. Math. Comput. 5 (1998) 457-463.

[5] S. Abbasbandy, Interpolation of fuzzy data by complete splines, J. Appl. Math. Comput. 8 (2001) 587-594.

[6] S. Abbasbandy, M. Amirfakhrian, Numerical approximation of fuzzy functions by fuzzy polynomials, Applied, Mathematics and computation 174 (2006) 1001-1006.

[7] S. Abbasbandy, M. Amirfakhrian, A new approach to universal appriximation of fuzzy functions on a discrete set of points, Applied Mathematical Modelling 30 (2006) 1525-1534.

[8] J. Gati, B. Bede, Spline appriximation of fuzzy functions, International conference on Applied Mathematics, (2005), 194-199.

[9] D. Dubois, H. Prade, Fuzzy sets and Systems:Theory and Application, Academic Press, New York, 1980.