Some L^p inequalities concerning the rate of growth of polynomials

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Abstract

In this paper we establish some L^p inequalities for polynomials with restricted zeros. Our results constitute multifarious generalizations which besides yielding several interesting results as corollaries also lead to some striking conclusions giving extensions and refinements of some known polynomial inequalities.

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1 Introduction and Statement of results

Let $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree atmost n and $P'(z)$ its derivative, then

$$Max_{|z|=1}|P'(z)| \leq nMax_{|z|=1}|P(z)| \tag{1.1}$$

and for every $p \geq 1$,

$$\left\{\int\limits_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq n\left\{\int\limits_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{1.2}$$

Inequality (1.1) is a classical result of Bernstein [18, 22] whereas inequality (1.2) is due to Zygmund [23]. Since inequality (1.2) was deduced from M. Riesz's interpolation formula [21] by means of Minkowski's inequality, it was not clear, whether the restriction on p was indeed essential. Finally it was proved by Arestov [2] that (1.2) remains true for $0 < p < 1$ as well.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then both the inequalities (1.1) and (1.2) can be sharpened. In fact, if $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$Max_{|z|=1}|P'(z)| \leq \frac{n}{2}Max_{|z|=1}|P(z)| \tag{1.3}$$

and

$$\left\{\int\limits_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq nC_p\left\{\int\limits_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{1.4}$$

where

$$C_p = \left\{\frac{1}{2\pi}\int\limits_0^{2\pi} \left|1 + e^{i\alpha}\right|^p d\alpha\right\}^{\frac{-1}{p}}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [16] whereas inequality (1.4) was found out by De-Bruijn [10] for $p \geq 1$. Rahman and Schmeisser [20] have shown that (1.4) holds for $0 < p < 1$ also. If we let $p \to \infty$ in (1.4), we get (1.3).

As a generalization of (1.3) Malik [17] proved that if $P(z) \neq 0$ in $|z| < k, k \geq 1$, then

$$Max_{|z|=1}|P'(z)| \leq \frac{n}{1+k} Max_{|z|=1}|P(z)| \tag{1.5}$$

whereas under the same hypothesis, Govil and Rahman [13] extended inequality (1.4) by showing that

$$\left\{ \int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta \right\}^{\frac{1}{p}} \leq nE_p \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \right\}^{\frac{1}{p}} \tag{1.6}$$

where

$$E_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|k + e^{i\alpha}\right|^p d\alpha \right\}^{\frac{-1}{p}}, \quad p \geq 1.$$

It was shown by Gardner and Weems [12] that inequality (1.6) also holds for $0 < p < 1$.

Chan and Malik [9] generalized (1.5) in a different direction and proved that, if $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then

$$Max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^t} Max_{|z|=1}|P(z)|. \tag{1.7}$$

Inequality (1.7) was independently proved by Qazi [19, *Lemma* 1], who also under the same hypothesis proved that

$$Max_{|z|=1}|P'(z)| \leq \frac{n}{1+S_1} Max_{|z|=1}|P(z)| \tag{1.8}$$

where

$$S_1 = k^{t+1} \left(\frac{\left(\frac{t}{n}\right)\left|\frac{a_t}{a_0}\right| k^{t-1} + 1}{\left(\frac{t}{n}\right)\left|\frac{a_t}{a_0}\right| k^{t+1} + 1} \right) \tag{1.9}$$

If $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu \neq 0$ in $|z| < k, k \geq 1$, then $\frac{t}{n}\left|\frac{a_t}{a_0}\right| k^t \leq 1$, which can also be taken as equivalent to $S_1 \geq k^t$. Hence inequality (1.8) is an improvement of inequality (1.7).

Aziz and Shah [6] investigated the dependence of $Max_{|z|=1}|P(Rz)-P(z)|$ on $Max_{|z|=1}|P(z)|$ and proved that if $P(z)=a_0+\sum\limits_{\nu=t}^{n}a_\nu z^\nu, t\geq 1$ is a polynomial of degree n, $P(z)\neq 0$ in $|z|<k, k\geq 1$, then for every $R>1$ and $|z|=1$,

$$|P(Rz)-P(z)|\leq\left\{\frac{R^n-1}{1+\psi_1(R)}\right\}Max_{|z|=1}|P(z)| \tag{1.10}$$

where

$$\psi_1(R)=k^{t+1}\left(\frac{\left(\frac{R^t-1}{R^n-1}\right)\left|\frac{a_t}{a_0}\right|k^{t-1}+1}{\left(\frac{R^t-1}{R^n-1}\right)\left|\frac{a_t}{a_0}\right|k^{t+1}+1}\right) \tag{1.11}$$

If we divide the two sides of (1.10) by $R-1$, make $R\to 1$ and noting that $\psi_1(R)\to S_1$ as $R\to 1$, we get (1.8).

Inequality (1.10) was generalized by Mir, Dewan and Singh [1] and proved that if $P(z)=a_0+\sum\limits_{\nu=t}^{n}a_\nu z^\nu, t\geq 1$, is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<k, k\geq 1$ and $m=Min_{|z|=k}|P(z)|$, then for every $R>1$ and $|z|=1$,

$$|P(Rz)-P(z)|\leq\left(\frac{R^n-1}{1+\psi_0(R)}\right)\left\{Max_{|z|=1}|P(z)|-m\right\} \tag{1.12}$$

where

$$\psi_0(R)=k^{t+1}\left(\frac{\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|}{|a_0|-m}k^{t+1}+1}\right) \tag{1.13}$$

If we divide the two sides of (1.12) by $R-1$, make $R\to 1$, we get a result due to Gardner, Govil and Weems [11].

The following result which is due to Aziz and Aliya [4] is of independent interest, because it provides generalizations and refinements of the inequalities $(1.3), (1.5), (1.7), (1.8), (1.10)$ and (1.12).

Theorem A. If $P(z)=a_0+\sum\limits_{\nu=t}^{n}a_\nu z^\nu, t\geq 1$ is a polynomial of degree n, having no zeros in $|z| < k, k \geq 1$, then for every $R > r \geq 1, 0 \leq \mu \leq 1$ and $|z| = 1$,

$$|P(Rz) - P(rz)| \leq \left(\frac{R^n - r^n}{1+\psi_\mu(R)}\right)\left\{Max_{|z|=1}|P(z)| - \mu m\right\} \tag{1.14}$$

where

$$\psi_\mu(R) = k^{t+1}\left(\frac{\left(\frac{R^t - r^t}{R^n - r^n}\right)\frac{|a_t|}{|a_0| - \mu m}k^{t-1} + 1}{\left(\frac{R^t - r^t}{R^n - r^n}\right)\frac{|a_t|}{|a_0| - \mu m}k^{t+1} + 1}\right) \tag{1.15}$$

and $m = Min_{|z|=k}|P(z)|$.

As a generalization of inequality (1.10) to the L^p norm, Aziz and Shah [7] proved the following:

Theorem B. if $P(z) = a_0 + \sum\limits_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k, k \geq 1$, then for every $R > 1, p > 0$ and α real

$$\left\{\int_0^{2\pi} \left|P(Re^{i\theta}) - P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left(R^n - 1\right)A_p\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{1.16}$$

where

$$A_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\psi_1(R) + e^{i\alpha}\right|^p d\alpha\right\}^{\frac{-1}{p}} \tag{1.17}$$

and $\psi_1(R)$ is as defined by formula (1.11).

In this paper we first prove the following more general result analogous to Theorem B, which not only generalize inequalities (1.12) and (1.15) to the L^p norm of $P(z)$ for every $p > 0$ but also leads to some striking conclusions giving refinements and generalizations of other well known results.

Theorem 1. If $P(z) = a_0 + \sum\limits_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$ and $m = Min_{|z|=k}|P(z)|$, then for every complex number β with $|\beta| \leq 1, p > 0, R > r \geq 1$ and α real,

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta}) - P(re^{i\theta}) + \left(\frac{R^n - r^n}{1+\psi_\mu(R)}\right)\mu m\beta\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left(R^n - r^n\right)B_p\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{1.18}$$

where

$$B_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|\psi_\mu(R) + e^{i\alpha}\right|^p d\alpha \right\}^{\frac{-1}{p}} \tag{1.19}$$

and $\psi_\mu(R)$ is as defined by formula (1.15).
If we let $p \to \infty$ in (1.18), noting that $B_p \to \frac{1}{1+\psi_\mu(R)}$ and choose argument of β with $|\beta| = 1$ suitably, we get (1.14).

The following corollary which is a generalization of (1.12) is obtained by taking $\mu = r = 1$ in Theorem 1.

Corollary 1. If $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$ and $m = Min_{|z|=k}|P(z)|$, then for every complex number β with $|\beta| \leq 1, p > 0, R > 1$ and α real,

$$\left\{ \int_0^{2\pi} \left|P(Re^{i\theta}) - P(e^{i\theta}) + \Big(\frac{R^n - 1}{1 + \psi_0(R)}\Big) m\beta\right|^p d\theta \right\}^{\frac{1}{p}} \leq \big(R^n - 1\big) C_p \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \right\}^{\frac{1}{p}} \tag{1.20}$$

where

$$C_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|\psi_0(R) + e^{i\alpha}\right|^p d\alpha \right\}^{\frac{-1}{p}} \tag{1.21}$$

and $\psi_0(R)$ is as defined by formula (1.13).
If we let $p \to \infty$ in (1.20), noting that $C_p \to \frac{1}{\psi_0(R)+1}$ and choose argument of β with $|\beta| = 1$ suitably, we get (1.12).

If we divide the two sides of inequality (1.18) by $R - r$ and make $R \to r$, we get the following interesting result.

Corollary 2. If $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$ and $m = Min_{|z|=k}|P(z)|$, then for every complex number β with $|\beta| \leq 1, p > 0, r \geq 1, 0 \leq \mu \leq 1$ and α real,

$$\left\{ \int_0^{2\pi} \left|e^{i\theta} P'(re^{i\theta}) + \frac{nr^{n-1}\mu}{1 + \psi_\mu(r)} m\beta\right|^p d\theta \right\}^{\frac{1}{p}} \leq nr^{n-1} B_p^* \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \right\}^{\frac{1}{p}} \tag{1.22}$$

where

$$\psi_\mu(r) = k^{t+1}\left(\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-\mu m}k^{t-1} + r^{n-t}}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-\mu m}k^{t+1} + r^{n-t}}\right) \tag{1.23}$$

and

$$B_p^* = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\psi_\mu(r) + e^{i\alpha}\right|^p d\alpha\right\}^{\frac{-1}{p}} \tag{1.24}$$

The following result which is an interesting generalization of inequality (1.8) and a refinement of (1.7) can be deduced from Corollary 2 by letting $p \to \infty$ and choose β with $|\beta| = 1$ suitably in (1.22).

Corollary 3. If $P(z) = a_0 + \sum\limits_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k, k \geq 1$ and $m = Min_{|z|=k}|P(z)|$, then for every $r \geq 1, 0 \leq \mu \leq 1$,

$$Max_{|z|=r\geq 1}|P'(z)| \leq \left(\frac{nr^{n-1}}{1+\psi_\mu(r)}\right)\left\{Max_{|z|=1}|P(z)| - \mu m\right\} \tag{1.25}$$

where $\psi_\mu(r)$ is given by (1.23).

If we take $\mu = 0$ and $t = 1$ in the above corollary 3, we get the following generalization of a result due to Govil, Rahman and Schemeisser [14].

Corollary 4. If $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then for every $e \geq 1$,

$$Max_{|z|=e\geq 1}|P'(z)| \leq ne^{n-1}\left(\frac{ne^{n-1}|a_0| + k^2|a_1|}{ne^{n-1}|a_0|(1+k^2) + 2k^2|a_1|}\right)Max_{|z|=1}|P(z)| \tag{1.26}$$

If we do not have the knowledge of $Min_{|z|=k}|P(z)|$, we obtain the following result which is a special case of Theorem 1.

Corollary 5. If $P(z) = a_0 + \sum\limits_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then for every $R > r \geq 1, p > 0$ and α real,

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta}) - P(re^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left(R^n - r^n\right)D_p\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{1.27}$$

where

$$D_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \psi_r(R) + e^{i\alpha} \right|^p d\alpha \right\}^{\frac{-1}{p}} \tag{1.28}$$

and

$$\psi_r(R) = k^{t+1} \left(\frac{\left(\frac{R^t - r^t}{R^n - r^n}\right) \left|\frac{a_t}{a_0}\right| k^{t-1} + 1}{\left(\frac{R^t - r^t}{R^n - r^n}\right) \left|\frac{a_t}{a_0}\right| k^{t+1} + 1} \right). \tag{1.29}$$

If we take $r = 1$ in the above corollary, we get Theorem B. Several other interesting results easily follows from Corollary 5. Here, we mention few of these. Since

$$\frac{R^t - r^t}{R^n - r^n} \leq \frac{t}{n} \tag{1.30}$$

holds for all $R > r \geq 1$ and $1 \leq t \leq n$ by considering the first derivative test for the function $\phi(R) = nR^t - tR^n$. Also if $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu \neq 0$ in $|z| < k, k \geq 1$, then $\frac{t}{n}\left|\frac{a_t}{a_0}\right| k^t \leq 1$, which when combined with (1.30) gives $\psi_r(R) \geq k^t$. Using this fact in inequality (1.28), we immediately get the following corollary.

Corollary 6. If $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then for every $R > r \geq 1, p > 0$ and α real,

$$\left\{ \int_0^{2\pi} \left| P(Re^{i\theta}) - P(re^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \leq \frac{\left(R^n - r^n\right)}{\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| k^t + e^{i\alpha} \right|^p d\alpha \right\}^{\frac{1}{p}}} \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \tag{1.31}$$

Clearly, Corollary 6 generalize inequality (1.7) and to obtain (1.7) from the above corollary, simply divide both sides of (1.31) by $R - 1$ (take $r = 1$) and let $R \to 1$.

Finally, as an application of Theorem 1, we prove the following generalization of a result of Aziz and Shah $[7, Theorem 2]$.

Theorem 2. If $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n

having no zeros in $|z| < k, k \geq 1$ and $m = Min_{|z|=k}|P(z)|$, then for every complex number β with $|\beta| \leq 1, p \geq 1, R > r \geq 1, 0 \leq \mu \leq 1$ and α real,

$$\left\{ \int_0^{2\pi} \left|P(Re^{i\theta}) + \Big(\frac{R^n - r^n}{1+\psi_\mu(R)}\Big)\mu m\beta\right|^p d\theta \right\}^{\frac{1}{p}} \leq \Big(r^n + (R^n - r^n)B_p\Big)\left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \right\}^{\frac{1}{p}} \tag{1.32}$$

where B_p and $\psi_\mu(R)$ are defined in Theorem 1.
If we let $p \to \infty$ in (1.32) and choose argument of β with $|\beta| = 1$ suitably, we get

$$Max_{|z|=R}|P(z)| \leq \left(\frac{R^n + r^n\psi_\mu(R)}{1+\psi_\mu(R)}\right) Max_{|z|=1}|P(z)| - \left(\frac{R^n - r^n}{1+\psi_\mu(R)}\right) m\mu. \tag{1.33}$$

It can be easily verified that for every n and $R > r \geq 1$, the function $\left(\frac{R^n+r^n x}{1+x}\right) Max_{|z|=1}|P(z)| - \left(\frac{R^n-r^n}{1+x}\right) m\mu$, is a non-decreasing function of x. If we combine this fact with Lemma 2 (stated in section 2), according to which $\psi_\mu(R) \geq k^t$ for $t \geq 1$, we get

$$Max_{|z|=R}|P(z)| \leq \left(\frac{R^n + r^n k^t}{1+k^t}\right) Max_{|z|=1}|P(z)| - \left(\frac{R^n - r^n}{1+k^t}\right) m\mu, \tag{1.34}$$

which is a generalization of a result due to Aziz [3, Theorem 4].
Remark 1. For $\mu = 0$ and $r = 1$, Theorem 2 reduces to a result of Aziz and Shah [6, Theorem 2].

2 LEMMAS

For the proof of these theorems we need the following lemmas.
Lemma 1. If $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$ and $q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then for $R > r \geq 1$ and $|z| = 1$,

$$\psi_\mu(R)\big|P(Rz) - P(rz)\big| \leq \big|q(Rz) - q(rz)\big| - (R^n - r^n)\mu m, \tag{2.1}$$

where $m = Min_{|z|=k}|P(z)|$ and $\psi_\mu(R)$ is defined by formula (1.15).

The above lemma is due to Aziz and Aliya [4].

Lemma 2. If $P(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu, t \geq 1$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$ and $m = Min_{|z|=k}|P(z)|$, then for $R > r \geq 1$ and $0 \leq \mu \leq 1$, we have

$$\psi_\mu(R) = k^{t+1}\left(\frac{\left(\frac{R^t-r^t}{R^n-r^n}\right)\frac{|a_t|}{|a_0|-\mu m}k^{t-1}+1}{\left(\frac{R^t-r^t}{R^n-r^n}\right)\frac{|a_t|}{|a_0|-\mu m}k^{t+1}+1}\right) \geq k^t.$$

Proof of Lemma 2. Since we have by an inequality (see [11, proof of lemma 3]),

$$\frac{|a_t|k^t}{|a_0| - m} \leq \frac{n}{t}, \quad t \geq 1. \tag{2.2}$$

Also for $0 \leq \mu \leq 1$, we have

$$\frac{|a_t|k^t}{|a_0| - \mu m} \leq \frac{|a_t|k^t}{|a_0| - m}. \tag{2.3}$$

Combining (2.2), (2.3) and (1.30), we get

$$\frac{|a_t|k^t}{|a_0| - \mu m} \leq \frac{R^n - r^n}{R^t - r^t}.$$

This inequality is clearly equivalent to

$$\left(\frac{R^t - r^t}{R^n - r^n}\right)\frac{|a_t|k^t}{|a_0| - \mu m}(k-1) \leq k - 1,$$

which implies

$$\left(\frac{R^t - r^t}{R^n - r^n}\right)\frac{|a_t|k^t}{|a_0| - \mu m} + 1 \leq \left(\frac{R^t - r^t}{R^n - r^n}\right)\frac{|a_t|k^t}{|a_0| - \mu m} + k,$$

from which Lemma 2 follows.

Lemma 3. If A, B and C are non-negative real numbers such that $B + C \leq A$, then for every real number α,

$$\left|(A - C)e^{i\alpha} + (B + C)\right| \leq \left|Ae^{i\alpha} + B\right|.$$

The above lemma is due to Aziz and Rather [5].
Lemma 4. If $P(z)$ is a polynomial of degree n, then for every $R \geq 1$ and $p > 0$,

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq R^n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}.$$

The above lemma is a simple consequences of a result of Hardy [15].
Lemma 5. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every $\beta, \gamma \in C$ with $|\beta| \leq 1, |\gamma| \leq 1$ and $R > r \geq 1,\ p > 0$ and α real

$$\begin{aligned}\int_0^{2\pi}\Bigg|\Bigg[P(Re^{i\theta}) - \beta P(re^{i\theta}) + \gamma\Bigg\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\Bigg\}P(re^{i\theta})\Bigg] \\ + e^{i\alpha}\Bigg[R^nP\Big(\frac{e^{i\theta}}{R}\Big) - \overline{\beta}r^nP\Big(\frac{e^{i\theta}}{r}\Big) + \overline{\gamma}r^n\Bigg\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\Bigg\}P\Big(\frac{e^{i\theta}}{r}\Big)\Bigg]\Bigg|^p d\theta \\ \leq \Bigg|R^n - \beta r^n + \gamma r^n\Bigg\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\Bigg\} \\ + e^{i\alpha}\Bigg[1 - \overline{\beta} + \overline{\gamma}\Bigg\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\Bigg\}\Bigg]\Bigg|^p \int_0^{2\pi}|P(e^{i\theta})|^p d\theta.\end{aligned}$$

The above lemma is due to Bidkham, Soleiman and Mir [8].

3 PROOF OF THEOREMS

Proof of Theorem 1. Since $P(z) \neq 0$ in $|z| < k, k \geq 1$, therefore by Lemma 1, for each $\theta, 0 \leq \theta < 2\pi, R > r \geq 1$ and $0 \leq \mu \leq 1$,

$$\psi_\mu(R)\left|P(Re^{i\theta}) - P(re^{i\theta})\right| \leq \left|R^nP\Big(\frac{e^{i\theta}}{R}\Big) - r^nP\Big(\frac{e^{i\theta}}{r}\Big)\right| - (R^n - r^n)\mu m,$$

which implies

$$\psi_\mu(R)\Big\{\big|P(Re^{i\theta})-P(re^{i\theta})\big|+\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m\Big\}$$
$$\leq \big|R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\big|-\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m. \quad (3.1)$$

Taking $A=\big|R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\big|$, $B=\big|P(Re^{i\theta})-P(re^{i\theta})\big|$ and $C=\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m$ in Lemma 5 and noting by Lemma 2 that $\psi_\mu(R)\geq k^t\geq 1$,

$$B+C\leq\psi_\mu(R)(B+C)\leq A-C\leq A,$$

we get for every real α,

$$\Bigg|\Big\{\big|R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\big|-\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m\Big\}e^{i\alpha}$$
$$+\Big\{\big|P(Re^{i\theta})-P(re^{i\theta})\big|+\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m\Big\}\Bigg|$$
$$\leq\Bigg|\big|R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\big|e^{i\alpha}+\big|P(Re^{i\theta})-P(re^{i\theta})\big|\Bigg|$$

This implies for each $p>0$,

$$\int_0^{2\pi}\Big|F(\theta)+e^{i\alpha}G(\theta)\Big|^p d\theta\leq\int_0^{2\pi}\Bigg|\big|R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\big|e^{i\alpha}+\big|P(Re^{i\theta})-P(re^{i\theta})\big|\Bigg|^p d\theta \quad (3.2)$$

where

$$F(\theta)=\big|P(Re^{i\theta})-P(re^{i\theta})\big|+\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m$$

and

$$G(\theta)=\big|R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\big|-\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m.$$

Integrating both sides of (3.2) with respect α from 0 to 2π, we get with the help of Lemma 5 (taking $\beta = 1$ and $\gamma = 0$) for each $p > 0,\ R > r \geq 1$,

$$\int_0^{2\pi}\int_0^{2\pi}\Big|F(\theta)+e^{i\alpha}G(\theta)\Big|^p d\theta d\alpha$$

$$\leq \int_0^{2\pi}\left\{\int_0^{2\pi}\Big|\big|R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\big|e^{i\alpha}+\big|P(Re^{i\theta})-P(re^{i\theta})\big|\Big|^p d\alpha\right\}d\theta$$

$$= \int_0^{2\pi}\left\{\int_0^{2\pi}\Big|\Big(R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\Big)e^{i\alpha}+\Big(P(Re^{i\theta})-P(re^{i\theta})\Big)\Big|^p d\alpha\right\}d\theta$$

$$= \int_0^{2\pi}\left\{\int_0^{2\pi}\Big|\Big(R^nP\big(\frac{e^{i\theta}}{R}\big)-r^nP\big(\frac{e^{i\theta}}{r}\big)\Big)e^{i\alpha}+\Big(P(Re^{i\theta})-P(re^{i\theta})\Big)\Big|^p d\theta\right\}d\alpha$$

$$\leq 2\pi\big(R^n-r^n\big)^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta. \tag{3.3}$$

Now for every real α and $t_1 \geq t_2 \geq 1$, we have

$$\Big|t_1+e^{i\alpha}\Big| \geq \Big|t_2+e^{i\alpha}\Big|,$$

which implies for every $p > 0$,

$$\int_0^{2\pi}\Big|t_1+e^{i\alpha}\Big|^p d\alpha \geq \int_0^{2\pi}\Big|t_2+e^{i\alpha}\Big|^p d\alpha,$$

If $F(\theta)\neq 0$, we take $t_1=\left|\frac{G(\theta)}{F(\theta)}\right|$ and $t_2=\psi_\mu(R)$, then from (3.1) and noting by Lemma 2 that $\psi_\mu(R)\geq 1$, we have $t_1\geq t_2\geq 1$, hence

$$\begin{aligned}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^p d\alpha &= \left|F(\theta)\right|^p\int_0^{2\pi}\left|1+\frac{G(\theta)}{F(\theta)}e^{i\alpha}\right|^p d\alpha\\ &= \left|F(\theta)\right|^p\int_0^{2\pi}\left|\frac{G(\theta)}{F(\theta)}+e^{i\alpha}\right|^p d\alpha\\ &= \left|F(\theta)\right|^p\int_0^{2\pi}\left|\left|\frac{G(\theta)}{F(\theta)}\right|+e^{i\alpha}\right|^p d\alpha\\ &\geq \left\{\left|P(Re^{i\theta})-P(re^{i\theta})\right|+\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m\right\}^p\int_0^{2\pi}\left|\psi_\mu(R)+e^{i\alpha}\right|^p d\alpha.\end{aligned}$$

If $F(\theta)=0$, then this inequality is trivially true. Using this in (3.3), we conclude that for each $p>0,\ R>r\geq 1$,

$$\begin{aligned}\int_0^{2\pi}\left|\psi_\mu(R)+e^{i\alpha}\right|^p d\alpha\int_0^{2\pi}&\left\{\left|P(Re^{i\theta})-P(re^{i\theta})\right|+\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\mu m\right\}^p d\theta\\ &\leq 2\pi\big(R^n-r^n\big)^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta. \qquad (3.4)\end{aligned}$$

Now using the fact that for every complex number β with $|\beta|\leq 1$,

$$\begin{aligned}&\left|P(Re^{i\theta})-P(re^{i\theta})+\beta\mu m\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big)\right|\\ &\qquad\leq \left|P(Re^{i\theta})-P(re^{i\theta})\right|+\mu m\Big(\frac{R^n-r^n}{1+\psi_\mu(R)}\Big),\end{aligned}$$

the desired result follows from (3.4).

Proof of Theorem 2. We have by Minkowski's inequality, for every $p\geq 1$

and $R > r \geq 1$,

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta}) + \beta\mu m\Big(\frac{R^n - r^n}{1+\psi_\mu(R)}\Big)\right|^p d\theta\right\}^{\frac{1}{p}}$$

$$= \left\{\int_0^{2\pi}\left|P(Re^{i\theta}) - P(re^{i\theta}) + \beta\mu m\Big(\frac{R^n - r^n}{1+\psi_\mu(R)}\Big) + P(re^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_0^{2\pi}\left|P(Re^{i\theta}) - P(re^{i\theta}) + \beta\mu m\Big(\frac{R^n - r^n}{1+\psi_\mu(R)}\Big)\right|^p d\theta\right\}^{\frac{1}{p}} + \left\{\int_0^{2\pi}\left|P(re^{i\theta})\right|^p\right\}^{\frac{1}{p}} \tag{3.5}$$

Inequality (3.52) in conjunction with inequality (1.18) of Theorem 1 and Lemma 4 gives for $R > r \geq 1,\ p \geq 1, |\beta| \leq 1$ and $0 \leq \mu \leq 1$,

$$\left\{\int_0^{2\pi}\left|P(Re^{i\theta}) + \Big(\frac{R^n - r^n}{1+\psi_\mu(R)}\Big)\mu m\beta\right|^p d\theta\right\}^{\frac{1}{p}}$$

$$\leq (R^n - r^n)B_p\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} + r^n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}$$

$$= \Big(r^n + (R^n - r^n)B_p\Big)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}$$

which is the inequality (1.32) and Theorem 2 is completely proved.

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