Approximate fixed point theorems for Geraghty-contractions

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Abstract

The purpose of this paper is to obtain necessary and sufficient conditions for existence approximate fixed point on Geraghty-contraction. In this paper, definitions of approximate -pair fixed point for two maps $T_\alpha, S_\alpha$, and their diameters are given in a metric space.

\textit{Key words:} Approximate fixed point; Approximate-pair fixed point; Geraghty-contraction.

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1 Introduction

In 1973, Geraghty [2] introduced the Geraghty-contraction and proved the fixed point property for it. In 2006, Mădălina Berinde [1] proved the approximate fixed point property for various types of well known generalized contractions on metric spaces.

In this paper, starting from the article of Zhang, Su, Cheng [3], we study Geraghty-contraction on partially ordered metric spaces, and we give some qualitative and quantitative results regarding approximate fixed points of such contraction mapping.

Throughout this article, we denote by $\Gamma$ the functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$  

**Definition 1.1** [2] Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a Geraghty-contraction if there exists $\beta \in \Gamma$ such that for any $x, y \in X$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

**Theorem 1.1** [2] Let $(X, d)$ be a complete metric space, and let $T : X \to X$ be an operator. Suppose that there exists $\beta \in \Gamma$ such that for any $x, y \in X$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Then $T$ has a unique fixed point.

In 2012, Caballero et al. considered another contraction condition also give a generalization of Theorem 1.2 by considering a non-self mapping, and they get the following theorems.

**Definition 1.2** [4] Let $A, B$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in \Gamma$ such that for any $x, y \in A$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$
In [1], the author defined the approximate fixed point property for self mapping on metric spaces.

**Definition 1.3** [1] Let \((X, d)\) be a metric space, \(\epsilon > 0\) and \(T : X \to X\) be a map. Then \(x_0 \in X\) is \(\epsilon\)-fixed point for \(T\) if \(d(Tx_0, x_0) < \epsilon\).

**Definition 1.4** [1] In this paper we will denote the set of all \(\epsilon\)-fixed points of \(T\), for a given \(\epsilon\), by:

\[
F_\epsilon(T) = \{ x \in X \mid x \text{ is an } \epsilon \text{-fixed point of } T \}.
\]

**Definition 1.5** [1] Let \((X, d)\) be a metric space and \(T : X \to X\) be a map. Then \(T\) has the approximate fixed point property if

\[
\forall \epsilon > 0, \ F_\epsilon(T) \neq \emptyset.
\]

**Definition 1.6** [5] Let \((X, \|\cdot\|)\) be a completely norm space and \(T : X \to X\), and \(T_\alpha : X \to X\) be a map as follow:

\[
T_\alpha = \alpha I + (1 - \alpha)T, \ 0 < \alpha < 1.
\]

Then \(x_0 \in X\) is \(\epsilon\)-fixed point for \(T_\alpha\) if \(\|T_\alpha x_0 - x_0\| < \epsilon\).

**Remark 1.1** [5] In this paper we will denote the set of all \(\epsilon\)-fixed points of \(T_\alpha\), for a given \(\epsilon\), by:

\[
F_\epsilon(T_\alpha) = \{ x \in X \mid x \text{ is an } \epsilon \text{-fixed point of } T_\alpha \}.
\]

### 2 \(\epsilon\)-fixed point in Geraghty-contraction for \(T\) and \(T_\alpha\) maps

In this section, we give some results on \(\epsilon\)-fixed point in Geraghty-contraction and its diameter.

**Theorem 2.1** Let \((X, d)\) be a metric space and \(T : X \to X\) be a map, \(x_0 \in X\) and \(\epsilon > 0\). If \(d(T^n(x_0), T^{n+k}(x_0)) \to 0\) as \(n \to \infty\) for some \(k > 0\), then \(T^k\) has an \(\epsilon\)-fixed point.
**Proof:** Since \(d(T^n(x_0), T^{n+k}(x_0))\) → 0 as \(n \to \infty\), \(\epsilon > 0\)

\[\exists n_0 > 0 \ s.t. \ \forall n \geq n_0 \ \ d(T^n(x_0), T^{n+k}(x_0)) < \epsilon.\]

Then

\[d(T^{n_0}(x_0), T^k(T^{n_0}(x_0)) < \epsilon,\]

therefore \(T^{n_0}(x_0)\) is an \(\epsilon\)-fixed point of \(T^k\).

**Theorem 2.2** Let \((X, d)\) be a metric space and \(T : X \to X\) a Geraghty-contraction map. Then:

\[\forall \epsilon > 0, \ F_\epsilon(T) \neq \emptyset.\]

**Proof:** Let \(\epsilon > 0\), \(x \in X\).

\[d(T^n(x), T^{n+1}(x)) = d(T(T^n-1(x)), T(T^n(x)))\]

\[\leq \beta(d(T^n-1(x), T^n(x)))d(T^n-1(x), T^n(x))\]

\[\leq \ldots.\]

\[\leq (\beta(d(T^n-1(x), T^n(x))))^{n-1}d(T(x), T^2(x))\]

\[\leq (\beta(d(T^n-1(x), T^n(x))))^n d(x, Tx).\]

But \(\beta \in \Gamma\) Therefore

\[\lim_{n \to \infty} d(T^n(x), T^{n+1}(x)) = 0, \ \forall x \in X.\]

Now by Theorem 2.3 it follows that \(F_\epsilon(T) \neq \emptyset, \forall \epsilon > 0.\)

**Theorem 2.3** Let \((X, d)\) be a metric space and \(T : X \to X\) a Geraghty-contraction map. If \(F_\epsilon(T)\), the set of Approximate fixed point of \(T\), is nonempty then the mapping

\[T_\alpha = \alpha I + (1 - \alpha)T, \ 0 < \alpha < 1\]

satisfy in Geraghty-contraction and \(F_\epsilon(T) = F_\epsilon(T_\alpha).\) Moreover \(d(T_\alpha^n(x), T_\alpha^{n+k}(x)) \to 0\) as \(n \to \infty\), for some \(k > 0, \epsilon > 0.\)
**Proof:** By the definition of $F_\epsilon(T)$, $F_\epsilon(T) = F_\epsilon(T_\alpha)$. Also, since $T$ satisfy in Geraghty-contraction and $I$ is identify function, it follows that $T_\alpha$ satisfy in Geraghty-contraction. Now, we prove $d(T_\alpha^n(x_0), T_\alpha^{n+k}(x_0)) \to 0$ as $n \to \infty$. Suppose $x \in X$ now, observe first that $d(T_\alpha^n(x), T_\alpha^{n+1}(x)) \leq \beta(d(x, T_\alpha x))d(x, T_\alpha x)$ and, by induction, that $d(T_\alpha^n(x), T_\alpha^{n+k}(x)) \leq (\beta(d(x, T_\alpha x)))^n d(x, T_\alpha x)$. Thus, for any $n$ and any $k > 0$, we have

$$d(T_\alpha^n(x), T_\alpha^{n+k}(x)) \leq \sum_{i=n}^{n+k-1} d(T_\alpha^i(x), T_\alpha^{i+1}(x))$$

$$\leq ((\beta(d(x, T_\alpha x)))^n + \cdots + (\beta(d(x, T_\alpha x)))^{n+k-1})d(x, T_\alpha x)$$

$$\leq \frac{1}{1 - (\beta(d(x, T_\alpha x)))} d(x, T_\alpha x).$$

But $\beta \in \Gamma$ Therefore $d(T_\alpha^n(x_0), T_\alpha^{n+k}(x_0)) \to 0$ as $n \to \infty$. $\blacksquare$

**Corollary 2.1** Let $(X, d)$ be a metric space and $T : X \to X$ a Geraghty-contraction map. If $F_\epsilon(T)$, the set of Approximate fixed point of $T$, is nonempty then the mapping

$$T_\alpha = \alpha I + (1 - \alpha)T, \ 0 < \alpha < 1$$

satisfy in Geraghty-contraction and $F_\epsilon(T) = F_\epsilon(T_\alpha)$. Then:

$$\forall \epsilon > 0, \ F_\epsilon(T_\alpha) \neq \emptyset.$$

**Proof:** By Theorem 2.4 it follows that $F_\epsilon(T) \neq \emptyset, \forall \epsilon > 0$, Therefore

$$F_\epsilon(T_\alpha) \neq \emptyset, \forall \epsilon > 0.$$

$\blacksquare$

**Definition 2.1** Let $T : X \to X$, be a map and $\epsilon > 0$. We define diameter $F_\epsilon(T)$ by

$$diam(F_\epsilon(T)) = \sup\{d(x, y) : \ x, y \in F_\epsilon(T)\}.$$
Theorem 2.4  Let $T : X \to X$, and $\epsilon > 0$. If $T : X \to X$ a Geraghty-contraction map. Then

$$ \text{diam}(F_{\epsilon}(T)) \leq \frac{2\epsilon}{1 - \beta(d(x, Tx))}, \quad \beta \in \Gamma. $$

Proof. If $x, y \in F_{\epsilon}(T)$, then

$$ d(x, y) \leq d(x, Tx) + d(Tx, Ty) + d(Ty, y) $$

$$ \leq \epsilon_1 + \beta(d(x, Tx))d(x, y) + \epsilon_2. $$

Put $\epsilon = \text{Max}\{\epsilon_1, \epsilon_2\}$, therefore $d(x, y) \leq \frac{2\epsilon}{1 - \beta(d(x, Tx))}$. Hence $\text{diam}(F_{\epsilon}(T)) \leq \frac{2\epsilon}{1 - \beta(d(x, Tx))}. \blacksquare$

Definition 2.2  Let $T : X \to X$ a map,

$$ T_\alpha = \alpha I + (1 - \alpha)T, \quad 0 < \alpha < 1 $$

a map and $\epsilon > 0$. We define diameter $F_{\epsilon}(T_\alpha)$ by

$$ \text{diam}(F_{\epsilon}(T_\alpha)) = \text{sup}\{d(x, y) : \ x, y \in F_{\epsilon}(T_\alpha)\}. $$

Theorem 2.5  Let $T : X \to X$, and $\epsilon > 0$. If $T : X \to X$ a Geraghty-contraction map and $T_\alpha : X \to X$ be a map as follow:

$$ T_\alpha = \alpha I + (1 - \alpha)T, \quad 0 < \alpha < 1 $$

Then

$$ \text{diam}(F_{\epsilon}(T_\alpha)) \leq \frac{2\epsilon}{1 - \beta(d(x, T_\alpha x))}. $$

Proof. If $x, y \in F_{\epsilon}(T_\alpha)$, then

$$ d(x, y) \leq d(x, T_\alpha x) + d(T_\alpha x, T_\alpha y) + d(T_\alpha y, y) $$

$$ \leq \epsilon_1 + \beta(d(x, T_\alpha x))d(x, y) + \epsilon_2. $$

Put $\epsilon = \text{Max}\{\epsilon_1, \epsilon_2\}$, therefore $d(x, y) \leq \frac{2\epsilon}{1 - \beta(d(x, T_\alpha x))}$. Hence $\text{diam}(F_{\epsilon}(T_\alpha)) \leq \frac{2\epsilon}{1 - \beta(d(x, T_\alpha x))}. \blacksquare$
3 Approximate-pair fixed point and \((T_\alpha, S_\alpha)\)

In this section we will consider the existence of approximate fixed points for two maps
\(T_\alpha : A \cup B \to A \cup B\), \(S_\alpha : A \cup B \to A \cup B\), where
\[
T_\alpha = \alpha I + (1 - \alpha)T, \quad S_\alpha = \alpha I + (1 - \alpha)S, \quad 0 < \alpha < 1,
\]
and \(T : A \cup B \to A \cup B\), \(S : A \cup B \to A \cup B\).

In 2011, Mohsenalhosseini et al. considered the existence of approximate best proximity points for two maps \(T : A \cup B \to A \cup B\), \(S : A \cup B \to A \cup B\) and they get the following theorems.

**Definition 3.1** [6] Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(T : A \cup B \to A \cup B\), \(S : A \cup B \to A \cup B\) be two maps such that \(T(A) \subseteq B\), \(S(B) \subseteq A\). A point \((x, y)\) in \(A \times B\) is said to be an approximate-pair fixed point for \((T, S)\) in \(X\), if there exists \(\epsilon > 0\)
\[
d(Tx, Sy) \leq d(A, B) + \epsilon.
\]
We say that the pair \((T, S)\) has the approximate-pair fixed property in \(X\) if
\[
P_{a(T, S)}(A, B) \neq \emptyset,
\]
where
\[
P_{a(T, S)}(A, B) = \{(x, y) \in A \times B : d(Tx, Sy) \leq d(A, B) + \epsilon \text{ for some } \epsilon > 0\}.
\]

**Theorem 3.1** [6] Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(T : A \cup B \to A \cup B\), \(S : A \cup B \to A \cup B\) be two maps such that \(T(A) \subseteq B\), \(S(B) \subseteq A\). If, for every \((x, y) \in A \times B\),
\[
d(T^n(x), S^n(y)) \to d(A, B)
\]
then \((T, S)\) has the approximate-pair fixed property.

**Definition 3.2** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and \(T_\alpha : A \cup B \to A \cup B\), \(S_\alpha : A \cup B \to A \cup B\) be two Geraghty-contraction maps such that \(T_\alpha(A) \subseteq B\), \(S_\alpha(B) \subseteq A\). A point \((x, y)\) in
$A \times B$ is said to be an approximate-pair fixed point for $(T_a, S_a)$ in $X$, if there exists $\epsilon > 0$

$$d(T_a x, S_a y) \leq d(A, B) + \epsilon.$$ We say that the pair $(T_a, S_a)$ has the approximate-pair fixed property in $X$ if

$$P_{(T_a, S_a)}^\epsilon(A, B) \neq \emptyset,$$

where

$$P_{(T_a, S_a)}^\epsilon(A, B) = \{(x, y) \in A \times B : d(T_a x, S_a y) \leq d(A, B) + \epsilon \text{ for some } \epsilon > 0\}.$$ Theorem 3.2 Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T_a : A \cup B \to A \cup B$, $S_a : A \cup B \to A \cup B$ be two maps such that $T_a(A) \subseteq B$, $S_a(B) \subseteq A$. If, for every $(x, y) \in A \times B$,

$$d(T_a^n(x), S_a^n(y)) \to d(A, B)$$

then $(T_a, S_a)$ has the approximate-pair fixed property.

Proof. For $\epsilon > 0$, Suppose $(x, y) \in A \times B$. Since

$$d(T_a^n(x), S_a^n(y)) \to d(A, B)$$

$$\exists n_0 > 0 \text{ s.t. } \forall n \geq n_0 : d(T_a^n(x), S_a^n(y)) < d(A, B) + \epsilon$$

Then $d(T_a(T_a^{n-1}(x), S_a(S_a^{n-1}(y)) < d(A, B) + \epsilon$ for every $n \geq n_0$. Put $x_0 = T_a^{n_0-1}(x)$ and $y_0 = S_a^{n_0-1}(y))$. Hence $d(T_a(x_0), S_a(y_0)) \leq d(A, B) + \epsilon$ and $P_{(T_a, S_a)}^\epsilon(A, B) \neq \emptyset$. 

Definition 3.3 Let $T_a : A \cup B \to A \cup B$, $S_a : A \cup B \to A \cup B$ be continuous maps such that $T_a(A) \subseteq B$, $S_a(B) \subseteq A$. We define diameter $P_{(T_a, S_a)}^\epsilon(A, B)$ by,

$$\text{diam}(P_{(T_a, S_a)}^\epsilon(A, B)) = \sup \{d(x, y) : d(T_a x, S_a y) \leq \epsilon + d(A, B) \text{ for some } \epsilon > 0\}.$$ Example 3.6. Suppose $A = \{(x, 0) : 0 \leq x \leq 1\}$, $B = \{(x, 1) : 0 \leq x \leq 1\}$, $T(x, 0) = T(x, 1) = (\frac{1}{2}, 1)$ and $S(x, 1) = S(x, 0) = (\frac{1}{2}, 0)$. Then $d(T(x, 0), S(y, 1)) = 1$ and $\text{diam}(P_{(T_a, S_a)}^\epsilon(A, B)) = \text{diam}(A \times B) = \sqrt{2}.$
**Theorem 3.3** Let $T_\alpha : A \cup B \to A \cup B$, $S_\alpha : A \cup B \to A \cup B$ be continuous maps such that $T_\alpha(A) \subseteq B$, $S_\alpha(B) \subseteq A$. If, there exists a $k \in [0,1]$, 
\[d(x, T_\alpha x) + d(S_\alpha y, y) \leq kd(x, y).\]
Then
\[
diam(P^{e \epsilon}_{(T_\alpha, S_\alpha)}(A, B)) \leq \frac{\epsilon}{1-k} + \frac{d(A, B)}{1-k} \text{ for some } \epsilon > 0.
\]

**Proof.** If $(x, y) \in P^{e \epsilon}_{(T_\alpha, S_\alpha)}(A, B)$, then
\[
d(x, y) \leq d(x, T_\alpha x) + d(T_\alpha x, S_\alpha y) + d(S_\alpha y, y) \leq \epsilon + kd(x, y) + d(A, B).
\]
Therefore $d(x, y) \leq \frac{\epsilon}{1-k} + \frac{d(A, B)}{1-k}$. Then $diam(P^{a \epsilon}_{(T_\alpha, S_\alpha)}(A, B)) \leq \frac{\epsilon}{1-k} + \frac{d(A, B)}{1-k}$.

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**References**


