Viscosity Approximation Methods for W-mappings in Hilbert space

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Abstract

We suggest a explicit viscosity iterative algorithm for finding a common element of the set of common fixed points for W-mappings which solves some variational inequality. Also, we prove a strong convergence theorem with some control conditions. Finally, examples and numerical results are also given.

Key words: Nonexpansive mapping, equilibrium problems, strongly positive linear bounded operator, fixed point, Hilbert space, W-mapping.
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1 Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. A mapping T of C into itself is nonexpansive if

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$\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$, we denote F(T) the set of fix points of T. The strong(weak) convergence of $\{x_n\}$ to x is written by $x_n \to x$ ($x_n \rightharpoonup x$) as $n \to \infty$.
For any $x \in H$, there exists a unique nearest pointin C, denoted it by $P_C(x)$ such that

$$\|x - P_C x\| \leq \|x - y\|, \text{ for all } y \in C,$$

such that a mapping P_C from H onto C is called the *metric projection*.
Recall that H satisfies the Opial's condition [6] if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$. A self mapping $f : C \to C$ is a contraction if there exists $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for each $x, y \in C$.
An operator A is said to be a strongly positive linear bounded operator on H, if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \text{ for all } x \in H.$$

Let F be a bifunction of $C \times C$ into R. The equilibrium problems for $C \times C \to C$, is to find $x \in C$ such that

$$F(x, y) \geq 0, \text{ for all } y \in C. \tag{1.1}$$

The set of solution of Eq.(1.1) is denoted by $EP(F)$. Several problems in physics, optimization, and economics reduce to find a solution of Eq.(1.1) [1], [4]. We consider the following iteration [10]

$$\begin{aligned}
U_{n,n+1} &:= I,\\
U_{n,n} &:= \lambda_n S_n U_{n,n+1} + (1 - \lambda_n)I,\\
U_{n,n-1} &:= \lambda_{n-1} S_{n-1} U_{n,n} + (1 - \lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &:= \lambda_k S_k U_{n,k+1} + (1 - \lambda_k)I,\\
&\vdots\\
U_{n,2} &:= \lambda_2 S_2 U_{n,3} + (1 - \lambda_2)I,\\
W_n = U_{n,1} &:= \lambda_1 S_1 U_{n,2} + (1 - \lambda_1)I,
\end{aligned} \tag{1.2}$$

where $\lambda_1, \lambda_2, \ldots$ are real numbers such that $0 \leq \lambda_n \leq 1$, and $S_1, S_2, \ldots$ be an infinite nonexpansive mappings. It is clear that nonexpansiviety of each S_n ensure the nonexapnsivity of W_n. Such a mapping W_n is called $W-$ mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In this paper, by intuition from [7], a new iterative scheme is introduced. This scheme find a common solution of the equilibrium problem (EP) and fixed point problem for an infinite family of nonexpansive mappings. Also, we prove a strong convergence theorem.
The following lemmas will be useful for proving the main results of this aticle:

Lemma 1.1 *[8] Let C be a nonempty closed convex subset of a Banach space E and $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. For any $n \geq 1$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strictly convex, then $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$.*

Lemma 1.2 *[8] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$ $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 1.3 *[8] Let C be a nonempty closed convex subset of stricly convex Banach E .Let $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then W is a nonexpansive mapping and $F(W) = \bigcap_{n=1}^{\infty} F(S_n)$.*

Lemma 1.4 *[2] Let C be a nonempty closed convex subset of a Hilbert space H and $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. If $K$is any bounded subset of C, then* $\limsup_{n\to\infty} \|Wx - W_n x\| = 0$.

Lemma 1.5 *[5] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq I - \rho\overline{\gamma}$.*

Lemma 1.6 *[9] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 1.7 *[2] Let H be a real Hilbert space. Then the following holds:*

(a) $\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle$ *for all* $x, y \in H$,
(b) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$,
(c) $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$.

Lemma 1.8 *[1] Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into R satisfying the following conditions:*

(A1) $F(x,x) = 0$ *for all* $x \in K$,
(A2) *F is monotone, that is,* $F(x,y) + F(y,x) \leq 0$ *for all* $x, y \in K$,
(A3) *for each* $x, y, z \in K$

$$\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y),$$

(A4) *for each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.Let $r > 0$ and $x \in H$.*
Then, there exists $z \in K$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K.$$

Lemma 1.9 *[3] Let K be a nonempty closed convex subset of H and let F be a bifunction of $K \times K$ into R satisfying $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to K$ as follows:*

$$T_r(x) = \{z \in K : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K\},$$

for all $x \in H$. Then the following hold

(i) T_r *is single valued map,*
(ii) T_r *is firmly nonexpansive, that is , for any* $x, y \in H$

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle,$$

(iii) $F(T_r) = EP(F)$,
(iv) $EP(F)$ *is closed and convex.*

Lemma 1.10 *[11] Assume* $\{a_n\}$ *be a sequence of nonnegative numbers such that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \delta_n,$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in real number such that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty;$
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty,$

then $\lim_{n\to\infty} a_n = 0.$

2 A Viscosity Iterative Algorithm

In this section, a new iterative scheme for finding a common element of the set of solutions for a equilibrium problems and the set of common fixed point for an infinite family of mappings in Hilbert space, is introduced.

Theorem 2.1 *Let*

- C *be a nonempty closed convex subset of a real Hilbert space* H,
- f *be a* $\rho-$*contractive map on* C,
- $J = \{1, 2, \ldots, k\}$ *be a finite index set,*
- *For each* $i \in J$, *let* G_i *be a bifunction from* $C \times C$ *into* R *satisfying* $(A1) - (A4)$,

- *A be a strongly positive linear bounded operator on H with coefficient $\varpi > 0$,*
- *$\{S_n\} : H \to H$ be a family of infinite nonexpansive mappings,*
- *$\bigcap_{i=1}^{k} F(W) \bigcap EP(G_i) \neq \emptyset$ where $F(W) = \bigcap_{j=1}^{n} F(S_j)$,*
- *$\{x_n\}$ be the sequence generated as following :*

$$\begin{cases} G_1(u_{n,1}, y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \\ G_2(u_{n,2}, y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \\ \vdots \\ G_k(u_{n,k}, y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \\ \theta_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \beta_n \gamma f(\theta_n) + (I - \beta_n A)\theta_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) W_n y_n, \end{cases}$$

where $\{W_n\}$ is a sequence defined by Eq.(1.2). Also, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $r_n \subset (0, \infty)$ and $0 < \gamma < \frac{\varpi}{\rho}$.

Suppose

(C1): $\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty,$
(C2): $\liminf_{n\to\infty} r_n > 0, \ \lim_{n\to\infty}(r_{n+1} - r_n) = 0;$
(C3): $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1,$
(C4): for each $i = 1, 2, \ldots, k \quad 0 < \lambda_i \leq c < 1.$

Then

(i) the sequence $\{x_n\}$ is bounded.
(ii) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$
(iii) $\lim_{n\to\infty} \|W_n y_n - y_n\| = 0.$

Proof. From (C1), we may assume that $\beta_n \leq \|A\|^{-1}$ for all $n \geq 1$. By Lemma 1.5, we obtain $\|I - \beta_n A\| \leq 1 - \beta_n \varpi$. It is clear that $P_{\bigcap_{i=1}^{k} F(W) \bigcap EP(G_i)}(I - A - \gamma f)$ is a contraction of C into itself. Indeed, for all $x, y \in C$

$$\begin{aligned}\|P_{\bigcap_{i=1}^k F(W)\bigcap EP(G_i)}(x) - & \ P_{\bigcap_{i=1}^k F(W)\bigcap EP(G_i)}(y)\| \\ & \leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\ & \leq \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\ & \leq (1 - \varpi)\|x - y\| + \gamma\rho\|x - y\| \\ & = (1 - (\varpi - \gamma\rho))\|x - y\|.\end{aligned}$$

(i): Let $x^* \in \bigcap_{i=1}^k F(W) \bigcap EP(G_i)$. Since $u_{n,i} = T_{r_n,i}x_n$ and $x^* = T_{r_n,i}x^*$, we see for any $n \geq N$

$$\|u_{n,i} - x^*\| = \|T_{r_n,i}x_n - T_{r_n,i}x^* \leq \|x_n - x^*\|\|,$$

thus

$$\|\theta_n - x^*\| \leq \|x_n - x^*\|. \tag{2.1}$$

Since f is $\rho-$contraction, we have

$$\begin{aligned}\|y_n - x^*\| &= \|\beta_n\gamma f(\theta_n) + (I - \beta_n A)\theta_n - x^*\| \\ &= \|\beta_n(\gamma f(\theta_n) - Ax^*) + (I - \beta_n A)(\theta_n - x^*)\| \\ &\leq \beta_n\|\gamma f(\theta_n) - Ax^*\| + \|I - \beta_n A\|\|\theta_n - x^*\| \\ &\leq \beta_n\gamma\|f(\theta_n) - f(x^*)\| + \beta_n\|\gamma f(x^*) - Ax^*\| \\ &+ (1 - \beta_n)\varpi\|x_n - x^*\|.\end{aligned}$$

From which it follows that

$$\|y_n - x^*\| \leq (1 - \beta_n(\varpi - \gamma\rho))\|x_n - x^*\| + \beta_n\|\gamma f(x^*) - Ax^*\|. \tag{2.2}$$

In viwe of Eq. (2.1) and Eq.(2.2), we obtain that

$$
\begin{aligned}
\|x_{n+1}-x^*\| &= \|\alpha_n x_n + (1-\alpha_n)W_n y_n - x^*\| \\
&= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(W_n y_n - x^*)\| \\
&\le \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|y_n - x^*\| \\
&\le \alpha_n\|x_n - x^*\| + (1-\alpha_n)\{(1-\beta_n(\varpi-\gamma\rho))\|x_n - x^*\| \\
&\quad + \beta_n\|\gamma f(x^*) - Ax^*\|\} \\
&= (1-\alpha_n)(1-\beta_n(\varpi-\gamma\rho))\|x_n - x^*\| \\
&\quad + \beta_n(\varpi-\gamma\rho)(1-\alpha_n)\frac{\|f(x^*) - Ax^*\|}{\varpi-\gamma\rho}.
\end{aligned}
$$

It follows by induction that

$$
\|x_{n+1}-x^*\| \le max\ \{\|x_1 - x^*\|, \frac{\|f(x^*) - Ax^*\|}{\varpi-\gamma\rho}\}.
$$

Therefore, the sequence $\{x_n\}$ is bounded and also $\{y_n\}$, $\{\theta_n\}$ are bounded.
(ii): Notic that

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|(I-\beta_{n+1}A)(\theta_{n+1}-\theta_n) + (\beta_n - \beta_{n+1})A\theta_n \\
&\quad + \gamma\{\beta_{n+1}(f(\theta_{n+1}) - f(\theta_n)) + f(\theta_n)(\beta_{n+1}-\beta_n)\}\| \\
&\le (1-\beta_{n+1}\varpi)\|\theta_{n+1}-\theta_n\| + |\beta_n - \beta_{n+1}|\|A\theta_n\| \\
&\quad + \gamma\beta_{n+1}\rho\|\theta_{n+1}-\theta_n\| + \gamma|\beta_{n+1}-\beta_n|\|f(\theta_n)\|.
\end{aligned}
$$

It follows that

$$
\|y_{n+1}-y_n\| \le (1-\beta_{n+1}(\varpi-\gamma\rho))\|\theta_{n+1}-\theta_n\| + |\beta_{n+1}-\beta_n|M. \quad (2.3)
$$

where $M = Sup\ _{n\ge 1}\ \{\|A\theta_n\| + \|\gamma(\theta_n)\|\}$.

Moreover, we have

$$
G_i(u_{n+1,i}, u_{n,i}) + \frac{1}{r_{n+1}}\langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1}\rangle \ge 0, \qquad (2.4)
$$

and

$$G_i(u_{n,i}, u_{n+1,i}) + \frac{1}{r_n}\langle u_{n+1,i} - u_{n,i}, u_{n,i} - x_n\rangle \geq 0. \tag{2.5}$$

Combining Eq.(2.4) and Eq.(2.5), we obtain

$$\begin{aligned}
0 &\leq r_{n+1}\{G_i(u_{n+1,i}, u_{n,i}) + G_i(u_{n,i}, u_{n+1,i})\} \\
&\quad + \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n}(u_{n,i} - x_n)\rangle \\
&\leq \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n}(u_{n,i} - x_n)\rangle,
\end{aligned}$$

from which it follows that

$$\langle u_{n,i} - u_{n+1,i}, u_{n,i} - u_{n+1,i} + x_{n+1} - x_n + x_n - u_{n,i} + \tfrac{r_{n+1}}{r_n}(u_{n,i} - x_n)\rangle \leq 0 \tag{2.6}$$

which implies that

$$\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n}\|x_n - u_{n,i}\|. \tag{2.7}$$

Using the condition (C2) and noting that there exists $b > 0$ such that $r_n > b > 0$, we obtain

$$\|\theta_{n+1} - \theta_n\| \leq \tfrac{1}{k}\sum_{i=1}^{k}\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n}\acute{M} \tag{2.8}$$

and

$$\|\theta_{n+1} - \theta_n\| \leq \|x_{n+1} - x_n\| + \frac{\acute{M}}{b}|r_{n+1} - r_n|, \tag{2.9}$$

where $\acute{M} := \frac{1}{k}\sum_{i=1}^{k}\|x_n - u_{n,i}\| < \infty$.
Moreover, we note that

$$\|W_{n+1}y_n - W_ny_n\|$$

$$
\begin{aligned}
&= \|\lambda_1 S_1 U_{n+1,2} y_n + (1-\lambda_1) y_n - (\lambda_1 S_1 U_{n,2} y_n + (1-\lambda_1) y_n)\| \\
&\le \lambda_1 \|U_{n+1,2} y_n - U_{n,2} y_n\| \\
&\le \lambda_1 \|\lambda_2 S_2 U_{n+1,3} y_n + (1-\lambda_2 y_n) - (\lambda_2 S_2 U_{n,3} y_n + (1-\lambda_2) y_n)\| \\
&\le \lambda_1 \lambda_2 \|U_{n+1,3} y_n - U_{n,3} y_n\| \\
&\quad \vdots \\
&\le (\prod_{m=1}^{n} \lambda_m) \|U_{n+1,n+1} y_n - U_{n,n+1} y_n\| \\
&= (\prod_{m=1}^{n} \lambda_m) \|\lambda_{n+1} S_{n+1} U_{n+1,n+2} y_n + (1-\lambda_{n+1} y_n - y_n\| \\
&= (\prod_{m=1}^{n} \lambda_m) \|\lambda_{n+1} S_{n+1} y_n - \lambda_{n+1}) y_n\|
\end{aligned}
$$

$$
= (\prod_{m=1}^{n+1} \lambda_m) \|S_{n+1} y_n - y_n\| \le \hat{M} (\prod_{m=1}^{n+1} \lambda_m) \tag{2.10}
$$

where $\hat{M} := Sup_{\ n \ge 1} \ \{\|S_{n+1} y_n - y_n\|\}$.
Combining Eq.(2.3), Eq.(2.9) and Eq.(2.10), we obtain

$$
\begin{aligned}
\|W_{n+1} y_{n+1} - W_n y_n\| &= \|W_{n+1} y_{n+1} - W_{n+1} y_n + W_{n+1} y_n - W_n y_n\| \\
&\le \|y_{n+1} - y_n\| + \|W_{n+1} y_n - W_n y_n\| \\
&\le \|\theta_{n+1} \theta_n\| + M|\beta_{n+1} - \beta_n| + \hat{M} (\prod_{m=1}^{n+1} \lambda_m) \\
&\le \|x_{n+1} - x_n\| + \frac{\acute{M}}{b} |r_{n+1} - r_n| + M|\beta_{n+1} - \beta_n| \\
&\qquad + \hat{M} (\prod_{m=1}^{n+1} \lambda_m).
\end{aligned}
$$

We have

$$
\limsup_{n \to \infty} (\|W_{n+1} y_{n+1} - W_n y_n\| - \|x_{n+1} - x_n\|) \le 0.
$$

From Lemma 1.6, we see that

$$\|W_n y_n - x_n\| \to 0 \quad , as \quad n \to \infty. \tag{2.11}$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\alpha_n)\|W_n y_n - x_n\| = 0$$

(iii): We shall prove that $\lim_{n\to\infty} \|x_n - z_n\| = 0$.
Notic that

$$\|u_{n,i} - x^*\|^2 \leq \langle T_{r_{n,i}} x_n - T_{r_{n,i}} x^*, x_n - x^* \rangle = \frac{1}{2}\{\|u_{n,i} - x^*\|^2 + \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2\}$$

thus

$$\|u_{n,i} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2. \tag{2.12}$$

From Eq.(2.12), we get

$$\|\theta_n - x^*\| = \|\sum_{i=1}^{k} \frac{1}{k}(u_{n,i} - x_n)\|^2 \leq \|x_n - x^*\|^2 - \frac{1}{k}\sum_{i=1}^{k} \|u_{n,i} - x_n\|^2. \tag{2.13}$$

It follows from Eq.(2.13) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(W_n y_n - W_n x^*)\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\{\|(I - \beta_n A)(\theta_n - x^*) \\
&\quad + \beta_n(\gamma f(\theta_n) - Ax^*)\|^2\} \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\{(1 - \beta_n \varpi)\|\theta_n - x^*\|^2 \\
&\quad + \beta_n \|\gamma f(\theta_n) - Ax^*)\|^2\} \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|\theta_n - x^*\|^2 \\
&\quad + \beta_n \|\gamma f(\theta_n) - Ax^*)\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\{\|x_n - x^*\|^2 \\
&\quad - \frac{1}{k}\sum_{i=1}^{k} \|u_{n,i} - x_n\|^2\} + \beta_n \|\gamma f(\theta_n) - Ax^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - (1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \\
&\quad + \beta_n \|\gamma f(\theta_n) - Ax^*)\|^2.
\end{aligned}$$

Thanks to the conditions of (C1)- (C4) and Eq.(2.12), we conclude that

$$\begin{aligned}(1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-x_n\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad +\beta_n\|\gamma f(\theta_n)-Ax^*)\|^2\\ &\le \|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|+\|x_n-x^*\|)\\ &\quad +\beta_n\|\gamma f(\theta_n)-Ax^*)\|^2\end{aligned}$$

$$\lim_{n\to\infty}\|u_{n,i}-x_n\|=0, \text{ for each } \ i=1,2,\ldots,k$$

also

$$\lim_{n\to\infty}\|\theta_n-x_n\|=\lim_{n\to\infty}\|u_{n,i}-x_n\|=0 \tag{2.14}$$

$$\|y_n-\theta_n\|=\beta_n\|\gamma f(\theta_n)-A\theta_n\|\to 0 \quad as \quad n\to\infty. \tag{2.15}$$

Moreover, we know that

$$\|y_n-x_n\|\le\|x_n-\theta_n\|+\|\theta_n-y_n\|$$

$$\|W_ny_n-y_n\|\le\|W_ny_n-x_n\|+\|x_n-\theta_n\|+\|\theta_n-y_n\|.$$

In viwe of Eq.(2.11), Eq.(2.14) and Eq.(2.15) , we can obtain

$$\lim_{n\to\infty}\|y_n-x_n\|=0 \tag{2.16}$$

$$\lim_{n\to\infty}\|W_ny_n-y_n\|=0. \tag{2.17}$$

Theorem 2.2 *Suppose all assumptions of Theorem 2.1 are holds. Then the sequence $\{x_n\}$ converge strongly to $\tilde{x}$, which solves the variational inequality*

$$\langle(A-\gamma f)\tilde{x},\tilde{x}-x_n\rangle\le 0, \quad \tilde{x}\in\bigcap_{i=1}^{k}F(W)\bigcap EP(G_i).$$

Equivalently, $P_{\bigcap_{i=1}^{k} F(W) \bigcap EP(G_i)}(I - A - \gamma f)(\tilde{x}) = \tilde{x}.$

Proof. We shall prove that

$$\limsup_{n\to\infty}\langle (A-\gamma f)x^*, y_n - x^*\rangle \leq 0,$$

where $x^* = P_{\bigcap_{i=1}^{k} F(W) \bigcap EP(G_i)} f(x^*)$.
We choose a subsequence $\{y_{n_p}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle (A-\gamma f)x^*, y_n - x^*\rangle = \lim_{p\to\infty}\langle (A-\gamma f)x^*, y_{n_p} - x^*\rangle, \tag{2.18}$$

since $\{y_{n_p}\}$ is bounded, there exits a subsequence of $\{y_{n_p}\}$, we denote it by $\{y_{n_p}\}$ such that $y_{n_p} \rightharpoonup q$, $q \in C$.

We shall show that $q \in \bigcap_{i=1}^{k} F(W) \bigcap EP(G_i)$. On the contrary, suppose that $q \notin F(W)$. By Opial's condition

$$\begin{aligned}
\liminf_{p\to\infty}\|y_{n_p} - q\| &< \liminf_{p\to\infty}\|y_{n_p} - Wq\| \\
&\leq \liminf_{p\to\infty}\{\|y_{n_p} - Wy_{n_p}\| + \|Wy_{n_p} - Wq\|\} \\
&\leq \liminf_{p\to\infty}\{\|y_{n_p} - Wy_{n_p}\| + \|y_{n_p} - q\|\}.
\end{aligned}$$

By virtue of Lemma 1.4 and noticing Eq.(2.17)

$$\begin{aligned}
\lim_{p\to\infty}\|Wy_{n_p} - y_{n_p}\| &\leq \lim_{p\to\infty}\{\|Wy_{n_p} - W_{n_p}y_{n_p}\| + \|W_{n_p}y_{n_p} - y_{n_p}\|\} \\
&\leq \lim_{p\to\infty}\{Sup_{\ x\in C}\ \ \|Wx - W_{n_p}x\|\} \\
&\quad + \lim_{p\to\infty}\|W_{n_p}y_{n_p} - y_{n_p}\| = 0.
\end{aligned}$$

It follows that

$$\liminf_{p\to\infty}\|y_{n_p} - q\| < \liminf_{p\to\infty}\|y_{n_p} - q\|.$$

This is a contradiction. Therefore, we have $q \in F(W)$. Also, we prove $q \in \bigcap_{i=1}^{k} EP(G_i)$.

For each $i \in J = \{1, 2, \ldots, k\}$, since $G_i(u_{n_p}, y) + \frac{1}{r_{n_p}} \langle y, u_{n_p} - x_{n_p} \rangle \geq 0$, from (A2), we see that

$$\frac{1}{r_{n_p}} \langle y - u_{n_p}, u_{n_p} - x_{n_p} \rangle \geq G_i(u_{n_p}, y) + G_i(y, u_{n_p}) \\ + \frac{1}{r_{n_p}} \langle y - u_{n_p}, u_{n_p} - x_{n_p} \rangle \geq G_i(y, u_{n_p}),$$

hence

$$\langle y - u_{n_p}, \frac{u_{n_p} - x_{n_p}}{r_{n_p}} \rangle \geq G_i(y, u_{n_p}), \text{ for all } y \in C.$$

Since $\frac{|u_{n_p} - x_{n_p}|}{r_{n_p}} \to 0$, $u_{n,i} \rightharpoonup q$, in viwe of (A4), we conclude

$$G_i(y, q) \leq 0, textbfforally \in C.$$

Let $0 < t \leq 1$, $y \in C$ and $y_t = ty + (1-t)q$. It is clear that $G_i(y_t, q) \leq 0$. From (A1)-(A4), we obtain

$$0 = G_i(y_t, y_t) \leq tG_i(y_t, y) + (1-t)G_i(y_t, q) \leq tG_i(y_t, y), \\ G_i(y, q) \geq 0, \forall y \in C.$$

Thus $q \in \bigcap_{i=1}^{k} EP(G_i)$.
From Eq.(2.18), we have

$$\limsup_{n\to\infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle = \lim_{p\to\infty} \langle (A - \gamma f)x^*, y_{n_p} - x^* \rangle \\ = \langle (A - \gamma f)x^*, x^* - q \rangle \leq 0.$$

It follows from Eq.(2.16) and Eq.(2.18) that

$$\limsup_{n\to\infty} \langle (A - \gamma f)x^*, x^* - x_n \rangle \leq \limsup_{n\to\infty} \langle (A - \gamma f)x^*, y_n - x_n \rangle \\ + \limsup_{n\to\infty} \langle (A - \gamma f)x^*, x^* - y_n \rangle \leq 0.$$

Finally, we prove that $x_n \to q$ where $x^* = P_{\bigcap_{i=1}^{k} F(W) \bigcap EP(G_i)} f(x^*)$.
By virtue of Lemma 1.7

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \|(I-\beta_n A)(\theta_n - x^*) + \beta_n(\gamma f(\theta_n) - Ax^*)\|^2 \\
&\le \|(I-\beta_n A)(\theta_n - x^*)\|^2 + 2\beta_n\langle \gamma f(\theta_n) - Ax^*, y_n - x^*\rangle \\
&\le \|(I-\beta_n A)(\theta_n - x^*)\|^2 \\
&\quad +2\beta_n\gamma\rho\|x_n - x^*\|\|y_n - x^*\| + 2\beta_n\langle \gamma f(\theta_n) - Ax^*, y_n - x^*\rangle \\
&\le (1-\beta_n\varpi)^2\|x_n - x^*\|^2 + \beta_n\gamma\rho(\|x_n - x^*\|^2 + \|y_n - x^*\|^2) \\
&\quad +2\beta_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle.
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\|y_n - x^*\|^2 &\le \frac{(1-\beta_n\varpi)^2 + \beta_n\gamma\rho}{1-\beta_n\gamma\rho}\|x_n - x^*\|^2 + \frac{2\beta_n}{1-\beta_n\gamma\rho}\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&\le \{1 - \frac{2\beta_n(\varpi-\gamma\rho)}{1-\beta_n\rho\gamma}\}\|x_n - x^*\|^2 \\
&\quad + \frac{2\beta_n(\varpi-\gamma\rho)}{1-\beta_n\gamma\rho}\{\frac{1}{\varpi-\gamma\rho}\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle + \frac{\beta_n\varpi^2}{2(\varpi-\gamma\rho)}L\},
\end{aligned}
$$

where $L = Sup\{\|x_n - x^*\|\}$.
Also

$$
\|x_{n+1} - x^*\|^2 \le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 \tag{2.19}
$$

it follows from Eq.(2.19) that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \le \{1 - (1-\alpha_n)&\frac{2\beta_n(\varpi-\gamma\rho)}{1-\beta_n\rho\gamma}\}\|x_n - x^*\|^2 \\
&+(1-\alpha_n)\frac{2\beta_n(\varpi-\gamma\rho)}{1-\beta_n\rho\gamma}\}\{\frac{1}{\varpi-\gamma\rho}\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&+\frac{\beta_n\varpi^2}{2(\varpi-\gamma\rho)}L\}.
\end{aligned}
$$

Let

$$\xi_n := (1-\alpha_n)\frac{2\beta_n(\varpi-\gamma\rho)}{1-\beta_n\rho\gamma}$$
$$\varepsilon_n := \frac{1}{\varpi-\gamma\rho}\langle\gamma f(x^*)-Ax^*, y_n-x^*\rangle + \frac{\beta_n\varpi^2}{2(\varpi-\gamma\rho)}L.$$

Therefore,

$$\|x_{n+1}-x^*\|^2 \leq (1-\xi_n)\|x_n-x^*\|^2 + \xi_n\varepsilon_n. \tag{2.20}$$

Thanks to the condition (C1) and Eq.(2.20), we conclude that

$$\lim_{n\to\infty}\xi_n = 0, \sum_{n=1}^{\infty}\xi_n = \infty.$$

From Lemma 1.10 we can obtain $x_n \to x^*$.

3 Numerical Example

In this section, we get one example is presented to grantee the main theorem (2.1).

Example 3.1 *Let $H=\mathbb{R}, C=[-1,1]$, $G_1(x,y)=-3x^2+xy+2y^2$, $G_2(x,y)=-4x^2+xy+3y^2$ and $G_3(x,y)=-9x^2+xy+8y^2$. Also, we consider $S_n=I, f(x)=\frac{x}{5}$ and $A=I$ be a strongly positive linear bounded operator with coefficient $\gamma=1$. It is easy to check that A and f satisfy all conditions in Theorem 2.2. For each $r>0$ and $x\in C$, there exists $z\in C$ such that, for any $y\in C$,*

$$G(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0$$
$$\Leftrightarrow -3z^2+zy+2y^2+\frac{1}{r}(y-z)(z-x)\geq 0$$
$$\Leftrightarrow 2ry^2+((r+1)z-x)y-3rz^2-z^2+zx\geq 0$$

Set $G(y)=2ry^2+((r+1)z-x)y-3rz^2-z^2+zx$. Then $G(y)$ is a quadratic function of y with coefficients $a=2r, b=(r+1)z-x$ and $c=-3rz^2-z^2+zx$. So

$$
\begin{aligned}
\Delta =& [(r+1)z-x]^2 - 8r(zx - z^2 - 3rz^2) \\
=& (r+1)^2z^2 - 2(r+1)xz + x^2 + 24r^2z^2 + 8rz^2 - 8rzx \\
=& x^2 - 2(5rz+z)x + (25r^2z^2 + 10rz^2 + z^2) \\
=& [(x-(5rz+z))]^2.
\end{aligned}
$$

Since $G(y) \geq 0$ for all $y \in C$, if and only if $\Delta = [(x-(5rz+z))]^2 \leq 0$. Therefore, $z = \frac{x}{5r+1}$, which yields $T_{r_n,1} = u_n^{(1)} = \frac{x_n}{5r_n+1}$. By the same argument, for G_2 and G_3, one can conclude $T_{r_n,2} = u_n^{(2)} \frac{x_n}{7r_n+1}, T_{r_n,3} = u_n^{(3)} = \frac{x_n}{17r_n+1}$. Let $r_n = \frac{n}{n+1}$. Hence

$$
\theta_n = \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)}}{3} = \frac{1}{3}\frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1}x_n.
$$

Suppose that $\alpha_n = \frac{2n-1}{10n-9}, \beta_n = \frac{1}{n}$ and $\lambda_n = \varepsilon$, we have

$$
\begin{aligned}
W_1 =& U_{11} = \lambda_1 S_1 U_{12} + (1-\lambda_1)I, \\
W_2 =& U_{21} = \lambda_1 S_1 U_{22} + (1-\lambda_1)I \\
=& \lambda_1 S_1\{\lambda_2 S_2 U_{23} + (1-\lambda_2)I\} + (1-\lambda_1)I, \\
=& \lambda_1\lambda_2 S_1 S_2 + \lambda_1(1-\lambda_2)S_1 + (1-\lambda_1)I, \\
W_3 =& U_{31} = \lambda_1 S_1 U_{32} + (1-\lambda_1)I \\
=& \lambda_1 S_1\{\lambda_2 S_2 U_{33} + (1-\lambda_2)I\} + (1-\lambda_1)I, \\
=& \lambda_1\lambda_2 S_1 S_2 U_{33} + \lambda_1(1-\lambda_2)S_1 + (1-\lambda_1)I, \\
=& \lambda_1\lambda_2 S_1 S_2\{\lambda_{33} S_3 U_{34} + (1-\lambda_3)I\} + \lambda_1(1-\lambda_2)S_1 + (1-\lambda_1)I, \\
=& \lambda_1\lambda_2\lambda_3 S_1 S_2 S_3 + \lambda_1\lambda_2(1-\lambda_3)S_1 S_2 + \lambda_1(1-\lambda_2)S_1 + (1-\lambda_1)I.
\end{aligned}
$$

By iteration this manner, we have

$$
\begin{aligned}
W_n = U_{n1} =& \lambda_1\lambda_2\cdots\lambda_n S_1 S_2\cdots S_n + \lambda_1\lambda_2\cdots\lambda_{n-1}(1-\lambda_n)S_1 S_2\cdots S_{n-1} \\
&+ \lambda_1\lambda_2\cdots\lambda_{n-2}(1-\lambda_{n-1})S_1 S_2\cdots S_{n-2} + \cdots + \lambda_1(1-\lambda_2)S_1 \\
&+ (1-\lambda_1)I.
\end{aligned}
$$

Let $T_n = I, \lambda_n = \varepsilon$, we obtain

$$
W_n = [\varepsilon^n + \varepsilon^{n-1}(1-\varepsilon) + \cdots + \varepsilon(1-\varepsilon) + (1-\varepsilon)]I = I.
$$

Hence

$$y_n = \left(\frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1}\right)\left(\frac{15n - 14}{15n}\right)x_n.$$

We have the following algorithm for the sequence $\{x_n\}$

$$x_{n+1} = \frac{2n-1}{10n-9}x_n + \frac{8n-8}{10n-9}y_n.$$

Choose $x_1 = 1$. By using MATLAB software, we obtain the following table and figure of the result.

n	x_n	n	x_n	n	x_n
1	1.0	11	0.003385691332	21	0.000002071711754
2	1.270565302	12	0.001637456001	22	0.0000009778618741
3	0.7706281483	13	0.0007885980277	23	0.0000004609899888
4	0.4251825949	14	0.0003784282172	24	0.0000002170787103
5	0.2242603285	15	0.0001810390789	25	0.0000001021162838
6	0.1151384546	16	0.00008637699279	26	0.00000004799105366
7	0.05805496956	17	0.00004111536174	27	0.00000002253432275
8	0.02889456779	18	0.00001953029496	28	0.00000001057245007
9	0.01424086637	19	0.000009260000453	29	0.000000004956540717
10	0.006965124578	20	0.000004383229046	30	0.000000002322073954

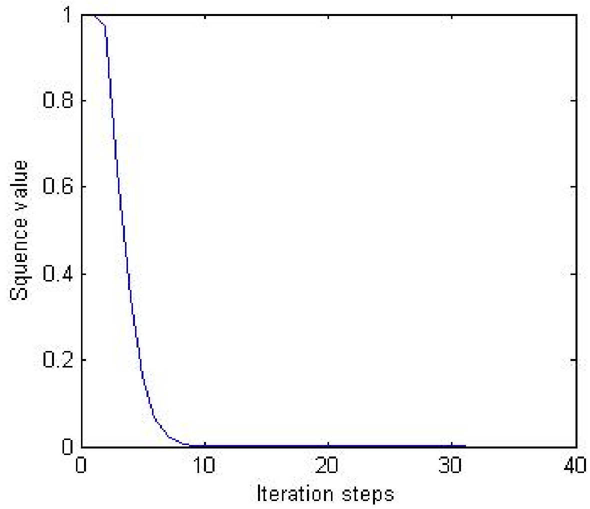

Fig. 1. The graph of $\{x_n\}$ with initial value $x_1 = 1$.

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