Viscosity Approximation Methods for W-mappings in Hilbert space

H. R. Sahebi\textsuperscript{a,*} S. Ebrahimi\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.
\textsuperscript{b}Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.

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Abstract

We suggest a explicit viscosity iterative algorithm for finding a common element of the set of common fixed points for W-mappings which solves some variational inequality. Also, we prove a strong convergence theorem with some control conditions. Finally, examples and numerical results are also given.

Key words: Nonexpansive mapping, equilibrium problems, strongly positive linear bounded operator, fixed point, Hilbert space, W-mapping.

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1 Introduction

Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. A mapping $T$ of $C$ into itself is nonexpansive if

\* Corresponding author’s E-mail: sahebi@aiau.ac.ir(S. Ebrahimi)
\[\|Tx - Ty\| \leq \|x - y\|\] for each \(x, y \in C\), we denote \(F(T)\) the set of fix points of \(T\). The strong(weak) convergence of \(\{x_n\}\) to \(x\) is written by \(x_n \to x (x_n \rightharpoonup x)\) as \(n \to \infty\).

For any \(x \in H\), there exists a unique nearest point in \(C\), denoted it by \(P_C(x)\) such that

\[\|x - P_C(x)\| \leq \|x - y\|,\] for all \(y \in C\),

such that a mapping \(P_C\) from \(H\) onto \(C\) is called the metric projection. Recall that \(H\) satisfies the Opial’s condition [6] if for any sequence \(\{x_n\}\) with \(x_n \to x\), the inequality

\[\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,\]

holds for every \(y \in H\) with \(x \neq y\). A self mapping \(f : C \to C\) is a contraction if there exists \(\alpha \in (0, 1)\) such that \(\|f(x) - f(y)\| \leq \alpha\|x - y\|\) for each \(x, y \in C\).

An operator \(A\) is said to be a strongly positive linear bounded operator on \(H\), if there is a constant \(\tilde{\gamma} > 0\) with property

\[\langle Ax, x \rangle \geq \tilde{\gamma}\|x\|^2,\] for all \(x \in H\).

Let \(F\) be a bifunction of \(C \times C\) into \(R\). The equilibrium problems for \(C \times C \to C\), is to find \(x \in C\) such that

\[F(x, y) \geq 0, \text{ for all } y \in C.\] (1.1)

The set of solution of Eq.(1.1) is denoted by \(EP(F)\). Several problems in physics, optimization, and economics reduce to find a solution of Eq.(1.1) [1], [4]. We consider the following iteration [10]

\[
\begin{align*}
U_{n,n+1} & := I, \\
U_{n,n} & := \lambda_n S_n U_{n,n+1} + (1 - \lambda_n)I, \\
U_{n,n-1} & := \lambda_{n-1} S_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\
& \vdots \\
U_{n,k} & := \lambda_k S_k U_{n,k+1} + (1 - \lambda_k)I, \\
& \vdots \\
U_{n,2} & := \lambda_2 S_2 U_{n,3} + (1 - \lambda_2)I, \\
W_n & = U_{n,1} = \lambda_1 S_1 U_{n,2} + (1 - \lambda_1)I,
\end{align*}
\] (1.2)

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where $\lambda_1, \lambda_2, \ldots$ are real numbers such that $0 \leq \lambda_n \leq 1$, and $S_1, S_2, \ldots$ be an infinite nonexpansive mappings. It is clear that nonexpansivity of each $S_n$ ensure the nonexapnsivity of $W_n$. Such a mapping $W_n$ is called $W$-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In this paper, by intuition from [7], a new iterative scheme is introduced. This scheme find a common solution of the equilibrium problem (EP) and fixed point problem for an infinite family of nonexpansive mappings. Also, we prove a strong convergence theorem.

The following lemmas will be useful for proving the main results of this article:

**Lemma 1.1** [8] Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. For any $n \geq 1$, let $W_n$ be the $W$-mapping of $C$ into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Then $W_n$ is asymptotically regular and nonexpansive. Further, if $E$ is strictly convex, then $F(W_n) = \cap_{i=1}^{n} F(S_i)$.

**Lemma 1.2** [8] Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1 \lim_{n \to \infty} U_{n,k,x}$ exists.

**Lemma 1.3** [8] Let $C$ be a nonempty closed convex subset of strictly convex Banach $E$. Let $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then $W$ is a nonexpansive mapping and $F(W) = \cap_{n=1}^{\infty} F(S_n)$.

**Lemma 1.4** [2] Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $\{S_n\} : C \to C$ be a family of infinitely nonexpansive mappings such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. If $K$ is any bounded subset of $C$, then $\limsup_{n \to \infty} \|Wx - W_nx\| = 0$. 
Lemma 1.5 [5] Assume $A$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\gamma > 0$ and $0 < \rho < ||A||^{-1}$. Then $||I - \rho A|| \leq I - \rho \gamma$.

Lemma 1.6 [9] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \beta_n \leq \limsup \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n \to \infty} (||y_n - y_n|| - ||x_n - x_n||) \leq 0$. Then $\lim_{n \to \infty} ||y_n - x_n|| = 0$.

Lemma 1.7 [2] Let $H$ be a real Hilbert space. Then the following holds:

(a) $||x + y||^2 \leq ||y||^2 + 2\langle x, x + y \rangle$ for all $x, y \in H$,
(b) $||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2$,
(c) $||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y \rangle$.

Lemma 1.8 [1] Let $K$ be a nonempty closed convex subset of $H$ and $F$ be a bi-function of $K \times K$ into $R$ satisfying the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$,
(A2) $F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$,
(A3) for each $x, y, z \in K$

$$\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y),$$

(A4) for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semi-continuous. Let $r > 0$ and $x \in H$.

Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in K.$$

Lemma 1.9 [3] Let $K$ be a nonempty closed convex subset of $H$ and let $F$ be a bifunction of $K \times K$ into $R$ satisfying (A1) – (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to K$ as follows:

$$T_r(x) = \{z \in K : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in K\},$$

for all $x \in H$. Then the following hold
(i) $T_r$ is single valued map.
(ii) $T_r$ is firmly nonexpansive, that is, for any $x, y \in H$

\[ \|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \]

(iii) $F(T_r) = EP(F)$.
(iv) $EP(F)$ is closed and convex.

Lemma 1.10 [11] Assume $\{a_n\}$ be a sequence of nonnegative numbers such that

\[ a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \]

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in real number such that

(i) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$,

then $\lim_{n \to \infty} a_n = 0$.

2 A Viscosity Iterative Algorithm

In this section, a new iterative scheme for finding a common element of the set of solutions for an equilibrium problems and the set of common fixed point for an infinite family of mappings in Hilbert space, is introduced.

Theorem 2.1 Let

- $C$ be a nonempty closed convex subset of a real Hilbert space $H$,
- $f$ be a $\rho$–contractive map on $C$,
- $J = \{1, 2, \ldots, k\}$ be a finite index set,
- For each $i \in J$, let $G_i$ be a bifunction from $C \times C$ into $R$ satisfying $(A1) - (A4)$,
A be a strongly positive linear bounded operator on $H$ with coefficient $\varpi > 0$,

- $\{S_n\} : H \to H$ be a family of infinite nonexpansive mappings,
- $\bigcap_{i=1}^{k} F(W) \cap EP(G_i) \neq \emptyset$ where $F(W) = \bigcap_{j=1}^{n} F(S_j)$,
- $\{x_n\}$ be the sequence generated as following :

$$
\begin{align*}
G_1(u_{n,1}, y) + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle & \geq 0, \\
G_2(u_{n,2}, y) + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle & \geq 0, \\
\vdots \\
G_k(u_{n,k}, y) + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle & \geq 0,
\end{align*}
$$

$$
\begin{align*}
\theta_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\
y_n = \beta_n \gamma f(\theta_n) + (I - \beta_n A) \theta_n, \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) W_n y_n,
\end{align*}
$$

where $\{W_n\}$ is a sequence defined by Eq.(1.2). Also, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $r_n \subset (0, \infty)$ and $0 < \gamma < \frac{\varpi}{p}$.

Suppose

(C1): $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,

(C2): $\liminf_{n \to \infty} r_n > 0$, $\lim_{n \to \infty} (r_{n+1} - r_n) = 0$;

(C3): $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$,

(C4): for each $i = 1, 2, \ldots, k$ $0 < \lambda_i \leq c < 1$.

Then

(i) the sequence $\{x_n\}$ is bounded.

(ii) $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

(iii) $\lim_{n \to \infty} \|W_n y_n - y_n\| = 0$.

Proof. From (C1), we may assume that $\beta_n \leq \|A\|^{-1}$ for all $n \geq 1$. By Lemma 1.5, we obtain $\|I - \beta_n A\| \leq 1 - \beta_n \varpi$. It is clear that $P \bigcap_{i=1}^{k} F(W) \cap EP(G_i) (I - A - \gamma f)$ is a contraction of $C$ into itself. Indeed,
for all $x, y \in C$

$$
\| P_{\bigcap_{i=1}^{k} F(W) \cap EP(G_i)}(x) - P_{\bigcap_{i=1}^{k} F(W) \cap EP(G_i)}(y) \|
\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\|
\leq \|I - A\| \|x - y\| + \gamma \|f(x) - f(y)\|
\leq (1 - \|\omega\|) \|x - y\| + \|f(x) - f(y)\|
= (1 - (\omega - \rho)) \|x - y\|.
$$

**(i):** Let $x^* \in \bigcap_{i=1}^{k} F(W) \cap EP(G_i)$. Since $u_{n,i} = T_{r_n,i} x_n$ and $x^* = T_{r_n,i} x^*$, we see for any $n \geq N$

$$
\| u_{n,i} - x^* \| = \| T_{r_n,i} x_n - T_{r_n,i} x^* \| \leq \| x_n - x^* \|,
$$
thus

$$
\| \theta_n - x^* \| \leq \| x_n - x^* \|. \quad (2.1)
$$

Since $f$ is $\rho-$contraction, we have

$$
\| y_n - x^* \| = \| \beta_n \gamma f(\theta_n) + (I - \beta_n A)\theta_n - x^* \|
= \| \beta_n (\gamma f(\theta_n) - Ax^*) + (I - \beta_n A)(\theta_n - x^*) \|
\leq \beta_n \gamma \| f(\theta_n) - Ax^* \| + \| I - \beta_n A \| \| \theta_n - x^* \|
\leq \beta_n \gamma \| f(\theta_n) - f(x^*) \| + \beta_n \| f(x^*) - Ax^* \|
+ (1 - \beta_n) \| x_n - x^* \|.
$$

From which it follows that

$$
\| y_n - x^* \| \leq (1 - \beta_n (\omega - \rho)) \| x_n - x^* \| + \beta_n \| f(x^*) - Ax^* \| . \quad (2.2)
$$

In viwe of Eq. (2.1) and Eq.(2.2), we obtain that
\[\|x_{n+1} - x^*\| = \|\alpha_n x_n + (1 - \alpha_n)W_n y_n - x^*\|\]
\[= \|\alpha_n (x_n - x^*) + (1 - \alpha_n)W_n y_n - x^*\|\]
\[\leq \alpha_n \|x_n - x^*\| + (1 - \alpha_n)\|y_n - x^*\|\]
\[\leq \alpha_n \|x_n - x^*\| + (1 - \alpha_n)(1 - \beta_n(\bar{\omega} - \gamma \rho))\|x_n - x^*\|\]
\[+ \beta_n \|\gamma f(x^*) - Ax^*\|\]
\[= (1 - \alpha_n)(1 - \beta_n(\bar{\omega} - \gamma \rho))\|x_n - x^*\|\]
\[+ \beta_n(\bar{\omega} - \gamma \rho)(1 - \alpha_n)\frac{\|f(x^*) - Ax^*\|}{\bar{\omega} - \gamma \rho}.\]

It follows by induction that

\[\|x_{n+1} - x^*\| \leq \max \{\|x_1 - x^*\|, \frac{\|f(x^*) - Ax^*\|}{\bar{\omega} - \gamma \rho}\}.\]

Therefore, the sequence \( \{x_n\} \) is bounded and also \( \{y_n\}, \{\theta_n\} \) are bounded.

(ii): Notic that

\[\|y_{n+1} - y_n\| = \|(I - \beta_n A) (\theta_{n+1} - \theta_n) + (\beta_n - \beta_{n+1}) A \theta_n\]
\[+ \gamma \{\beta_{n+1}(f(\theta_{n+1}) - f(\theta_n)) + f(\theta_n)(\beta_{n+1} - \beta_n)\}\|\]
\[\leq (1 - \beta_{n+1} \bar{\omega})\|\theta_{n+1} - \theta_n\| + |\beta_n - \beta_{n+1}|\|A \theta_n\|
\[+ \gamma \beta_{n+1} \rho\|\theta_{n+1} - \theta_n\| + \gamma |\beta_{n+1} - \beta_n|\|f(\theta_n)\|.\]

It follows that

\[\|y_{n+1} - y_n\| \leq (1 - \beta_{n+1}(\bar{\omega} - \gamma \rho))\|\theta_{n+1} - \theta_n\| + |\beta_{n+1} - \beta_n|M. (2.3)\]

where \( M = \text{Sup }_{n \geq 1} \{\|A \theta_n\| + \|\gamma (\theta_n)\|\}.\)

Moreover, we have

\[G^*_i(u_{n+1,i}, u_{n,i}) + \frac{1}{r_{n+1}} \langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} \rangle \geq 0, \quad (2.4)\]

and
\begin{equation}
G_i(u_{n,i}, u_{n+1,i}) + \frac{1}{r_n} (u_{n+1,i} - u_{n,i}, u_{n,i} - x_n) \geq 0. 
\end{equation}

Combining Eq.(2.4) and Eq.(2.5), we obtain

\begin{align*}
0 & \leq r_{n+1} \left\{ G_i(u_{n+1,i}, u_{n,i}) + G_i(u_{n,i}, u_{n+1,i}) \right\} \\
& + \left\langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n} (u_{n,i} - x_n) \right\rangle \\
& \leq \left\langle u_{n,i} - u_{n+1,i}, u_{n+1,i} - x_{n+1} - \frac{r_{n+1}}{r_n} (u_{n,i} - x_n) \right\rangle,
\end{align*}

from which it follows that

\begin{equation}
\langle u_{n,i} - u_{n+1,i}, u_{n,i} - u_{n+1,i} + x_{n+1} - x_n + x_n - u_{n,i} + \frac{r_{n+1}}{r_n} (u_{n,i} - x_n) \rangle \leq 0 
\end{equation}

which implies that

\begin{equation}
\| u_{n+1,i} - u_{n,i} \| \leq \| x_{n+1} - x_n \| + \frac{|r_{n+1} - r_n|}{r_n} \| x_n - u_{n,i} \|. 
\end{equation}

Using the condition (C2) and noting that there exists \( b > 0 \) such that \( r_n > b > 0 \), we obtain

\begin{equation}
\| \theta_{n+1} - \theta_n \| \leq \frac{1}{k} \sum_{i=1}^{k} \| u_{n+1,i} - u_{n,i} \| \leq \| x_{n+1} - x_n \| + \frac{|r_{n+1} - r_n|}{r_n} \tilde{M}
\end{equation}

and

\begin{equation}
\| \theta_{n+1} - \theta_n \| \leq \| x_{n+1} - x_n \| + \frac{\tilde{M}}{b} |r_{n+1} - r_n|,
\end{equation}

where \( \tilde{M} := \frac{1}{k} \sum_{i=1}^{k} \| x_n - u_{n,i} \| < \infty \).

Moreover, we note that

\begin{equation}
\| W_{n+1}y_n - W_n y_n \|
\end{equation}
\[ = \|\lambda_1 S_{1}U_{n+1,2}y_n + (1 - \lambda_1)y_n - (\lambda_1 S_{1}U_{n,2}y_n + (1 - \lambda_1)y_n)\| \\
\leq \lambda_1\|U_{n+1,2}y_n - U_{n,2}y_n\| \\
\leq \lambda_1\|\lambda_2 S_{2}U_{n+1,3}y_n + (1 - \lambda_2)y_n - (\lambda_2 S_{2}U_{n,3}y_n + (1 - \lambda_2)y_n)\| \\
\leq \lambda_1\lambda_2\|U_{n+1,3}y_n - U_{n,3}y_n\| \\
\vdots \\
\leq \left(\prod_{m=1}^{n}\lambda_m\right)\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\| \\
= \left(\prod_{m=1}^{n}\lambda_m\right)\|\lambda_{n+1}S_{n+1}U_{n+1,n+2}y_n + (1 - \lambda_{n+1}y_n - y_n)\| \\
= \left(\prod_{m=1}^{n}\lambda_m\right)\|\lambda_{n+1}S_{n+1}y_n - \lambda_{n+1})y_n\| \\
\]
\[ = \left(\prod_{m=1}^{n+1}\lambda_m\right)\|S_{n+1}y_n - y_n\| \leq \hat{M}\left(\prod_{m=1}^{n+1}\lambda_m\right) \tag{2.10} \]

where \(\hat{M} := \text{Sup}_{n \geq 1} \{\|S_{n+1}y_n - y_n\|\}\).

Combining Eq.(2.3), Eq.(2.9) and Eq.(2.10), we obtain

\[ \|W_{n+1}y_{n+1} - W_ny_n\| = \|W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_ny_n\| \\
\leq \|y_{n+1} - y_n\| + \|W_{n+1}y_n - W_ny_n\| \\
\leq \|\theta_{n+1}\theta_n\| + M|\beta_{n+1} - \beta_n| + \hat{M}\left(\prod_{m=1}^{n+1}\lambda_m\right) \\
\leq \|x_{n+1} - x_n\| + \frac{\hat{M}}{\beta_n}\|r_{n+1} - r_n\| + M|\beta_{n+1} - \beta_n| \\
+ \hat{M}\left(\prod_{m=1}^{n+1}\lambda_m\right). \]

We have

\[ \limsup_{n \to \infty}(\|W_{n+1}y_{n+1} - W_ny_n\| - \|x_{n+1} - x_n\|) \leq 0. \]

From Lemma 1.6, we see that
\[ \|W_n y_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \quad (2.11) \]

which implies that
\[ \lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \alpha_n)\|W_n y_n - x_n\| = 0 \]

(iii): We shall prove that \( \lim_{n \to \infty} \|x_n - z_n\| = 0 \).

Notic that
\[ \|u_{n,i} - x^*\|^2 \leq \langle T_{n,i} x_n - T_{n,i} x^*, x_n - x^* \rangle = \frac{1}{2} \{ \|u_{n,i} - x^*\|^2 + \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2 \} \]

thus
\[ \|u_{n,i} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|u_{n,i} - x_n\|^2. \quad (2.12) \]

From Eq.(2.12), we get
\[ \|\theta_n - x^*\| = \left\| \frac{1}{k} \sum_{i=1}^{k} (u_{n,i} - x_n) \right\|^2 \leq \|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2. \quad (2.13) \]

It follows from Eq.(2.13) that
\[
\|x_{n+1} - x^*\|^2 = \|\alpha_n (x_n - x^*) + (1 - \alpha_n) (W_n y_n - W_n x^*)\|^2 \\
\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \left\{ \left\| (I - \beta_n A) (\theta_n - x^*) \right\|^2 + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \right\} \\
\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \left\{ (1 - \beta_n \omega) \|\theta_n - x^*\|^2 + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \right\} \\
\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|\theta_n - x^*\|^2 + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \{\|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2\} + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\leq \|x_n - x^*\|^2 - (1 - \alpha_n) \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 + \beta_n \|\gamma f(\theta_n) - Ax^*\|^2.
\]
Thanks to the conditions of (C1)-(C4) and Eq.(2.12), we conclude that

\[
(1 - \alpha_n) \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
+ \beta_n \|\gamma f(\theta_n) - Ax^*\|^2 \\
\leq \|x_{n+1} - x_n\| (\|x_{n+1} - x^*\| + \|x_n - x^*\|) \\
+ \beta_n \|\gamma f(\theta_n) - Ax^*\|^2
\]

\[
\lim_{n \to \infty} \|u_{n,i} - x_n\| = 0, \text{ for each } i = 1, 2, \ldots, k
\]

also

\[
\lim_{n \to \infty} \|\theta_n - x_n\| = \lim_{n \to \infty} \|u_{n,i} - x_n\| = 0 \quad (2.14)
\]

\[
\|y_n - \theta_n\| = \beta_n \|\gamma f(\theta_n) - A\theta_n\| \to 0 \text{ as } n \to \infty. \quad (2.15)
\]

Moreover, we know that

\[
\|y_n - x_n\| \leq \|x_n - \theta_n\| + \|\theta_n - y_n\|
\]

\[
\|W_n y_n - y_n\| \leq \|W_n y_n - x_n\| + \|x_n - \theta_n\| + \|\theta_n - y_n\|
\]

In view of Eq.(2.11), Eq.(2.14) and Eq.(2.15), we can obtain

\[
\lim_{n \to \infty} \|y_n - x_n\| = 0 \quad (2.16)
\]

\[
\lim_{n \to \infty} \|W_n y_n - y_n\| = 0. \quad (2.17)
\]

**Theorem 2.2** Suppose all assumptions of Theorem 2.1 are holds. Then the sequence \(\{x_n\}\) converge strongly to \(\tilde{x}\), which solves the variational inequality

\[
\langle (A - \gamma f)\tilde{x}, \tilde{x} - x_n \rangle \leq 0, \quad \tilde{x} \in \bigcap_{i=1}^{k} F(W) \cap EP(G_i).
\]
Equivalently, \( P \bigcap_{i=1}^{k} F(W) \bigcap_{EP(G_i)} (I - A - \gamma f)(\tilde{x}) = \tilde{x} \).

**Proof.** We shall prove that
\[
\limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle \leq 0,
\]
where \( x^* = P \bigcap_{i=1}^{k} F(W) \bigcap_{EP(G_i)} f(x^*) \).

We choose a subsequence \( \{y_{n_p}\} \) of \( \{y_n\} \) such that
\[
\limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle = \lim_{p \to \infty} \langle (A - \gamma f)x^*, y_{n_p} - x^* \rangle, \tag{2.18}
\]
since \( \{y_{n_p}\} \) is bounded, there exits a subsequence of \( \{y_{n_p}\} \), we denote it by \( \{y_{n_p}\} \) such that \( y_{n_p} \rightharpoonup q, q \in C \).

We shall show that \( q \in \bigcap_{i=1}^{k} F(W) \bigcap_{EP(G_i)} \). On the contrary, suppose that \( q \notin F(W) \). By Opial’s condition
\[
\liminf_{p \to \infty} \|y_{n_p} - q\| < \liminf_{p \to \infty} \|y_{n_p} - Wq\| \leq \liminf_{p \to \infty} \{\|y_{n_p} - Wy_{n_p}\| + \|W y_{n_p} - Wq\|\} \\
\leq \liminf_{p \to \infty} \{\|y_{n_p} - W y_{n_p}\| + \|y_{n_p} - q\|\}.
\]

By virtue of Lemma 1.4 and noticing Eq.(2.17)
\[
\lim_{p \to \infty} \|W y_{n_p} - y_{n_p}\| \leq \lim_{p \to \infty} \{\|W y_{n_p} - W y_{n_p} y_{n_p}\| + \|W y_{n_p} y_{n_p} - y_{n_p}\|\} \\
\leq \lim_{p \to \infty} \{\sup_{x \in C} \|W x - W y_{n_p} x\|\} + \lim_{p \to \infty} \|W y_{n_p} y_{n_p} - y_{n_p}\| = 0.
\]

It follows that
\[
\liminf_{p \to \infty} \|y_{n_p} - q\| < \liminf_{p \to \infty} \|y_{n_p} - q\|.
\]

This is a contradiction. Therefore, we have \( q \in F(W) \). Also, we prove \( q \in \bigcap_{i=1}^{k} EP(G_i) \).
For each $i \in J = \{1, 2, \ldots, k\}$, since $G_i(u_n, y) + \frac{1}{r_n} \langle y, u_n - x_n \rangle \geq 0$, from (A2), we see that

$$
\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq G_i(u_n, y) + G_i(y, u_n)
$$

$$
+ \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq G_i(y, u_n),
$$

hence

$$
\langle y - u_n, \frac{u_n - x_n}{r_n} \rangle \geq G_i(y, u_n), \text{ for all } y \in C.
$$

Since $\frac{|u_n - x_n|}{r_n} \to 0$, $u_{n,i} \rightharpoonup q$, in view of (A4), we conclude

$$
G_i(y, q) \leq 0, \text{ for all } y \in C.
$$

Let $0 < t \leq 1$, $y \in C$ and $y_t = ty + (1 - t)q$. It is clear that $G_i(y_t, q) \leq 0$.

From (A1)-(A4), we obtain

$$
0 = G_i(y_t, y_t) \leq tG_i(y_t, y) + (1 - t)G_i(y_t, q) \leq tG_i(y_t, y),
$$

$$
G_i(y, q) \geq 0, \forall y \in C.
$$

Thus $q \in \bigcap_{i=1}^{k} EP(G_i)$.

From Eq.(2.18), we have

$$
\limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle = \lim_{p \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle
$$

$$
= \langle (A - \gamma f)x^*, x^* - q \rangle \leq 0.
$$

It follows from Eq.(2.16) and Eq.(2.18) that

$$
\limsup_{n \to \infty} \langle (A - \gamma f)x^*, x^* - x_n \rangle \leq \limsup_{n \to \infty} \langle (A - \gamma f)x^*, y_n - x^* \rangle
$$

$$
+ \limsup_{n \to \infty} \langle (A - \gamma f)x^*, x^* - y_n \rangle \leq 0.
$$

Finally, we prove that $x_n \to q$ where $x^* = P_{\bigcap_{i=1}^{k} F(W) \cap EP(G_i)} f(x^*)$.

By virtue of Lemma 1.7
\[ \|y_n - x^*\|^2 = \|(I - \beta_n A)(\theta_n - x^*) + \beta_n(\gamma f(\theta_n) - Ax^*)\|^2 \]
\[ \leq \|(I - \beta_n A)(\theta_n - x^*)\|^2 + 2\beta_n\langle \gamma f(\theta_n) - Ax^*, y_n - x^* \rangle \]
\[ \leq \|(I - \beta_n A)(\theta_n - x^*)\|^2 + 2\beta_n\gamma \rho \|x_n - x^*\| \|y_n - x^*\| + 2\beta_n \langle \gamma f(\theta_n) - Ax^*, y_n - x^* \rangle \]
\[ \leq (1 - \beta_n \omega)^2 \|x_n - x^*\|^2 + \beta_n \gamma \rho (\|x_n - x^*\|^2 + \|y_n - x^*\|^2) \]
\[ + 2\beta_n \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle. \]

from which it follows that

\[ \|y_n - x^*\|^2 \leq \frac{(1 - \beta_n \omega)^2 + \beta_n \gamma \rho}{1 - \beta_n \gamma \rho} \|x_n - x^*\|^2 + \frac{2\beta_n}{1 - \beta_n \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \]
\[ \leq \left\{ 1 - \frac{2\beta_n(\omega - \gamma \rho)}{1 - \beta_n \gamma \rho} \right\} \|x_n - x^*\|^2 \]
\[ + \frac{2\beta_n(\omega - \gamma \rho)}{1 - \beta_n \gamma \rho} \left\{ \frac{1}{\omega - \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle + \frac{\beta_n \omega^2}{2(\omega - \gamma \rho)} L \right\}, \]

where \( L = \text{Sup}\{\|x_n - x^*\|\} \).

Also

\[ \|x_{n+1} - x^*\|^2 \leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 \quad (2.19) \]

it follows from Eq.(2.19) that

\[ \|x_{n+1} - x^*\|^2 \leq \left\{ 1 - (1 - \alpha_n) \frac{2\beta_n(\omega - \gamma \rho)}{1 - \beta_n \gamma \rho} \right\} \|x_n - x^*\|^2 \]
\[ + (1 - \alpha_n) \frac{2\beta_n(\omega - \gamma \rho)}{1 - \beta_n \gamma \rho} \left\{ \frac{1}{\omega - \gamma \rho} \langle \gamma f(x^*) - Ax^*, y_n - x^* \rangle \right\} \]
\[ + \frac{\beta_n \omega^2}{2(\omega - \gamma \rho)} L \].

Let
\[ \xi_n := (1 - \alpha_n) \frac{2\beta_n(\varpi - \gamma \rho)}{1 - \beta_n \rho \gamma} \]
\[ \varepsilon_n := \frac{1}{\varpi - \gamma \rho} (\gamma f(x^*) - Ax^*, y_n - x^*) + \frac{\beta_n \varpi^2}{2(\varpi - \gamma \rho)} L. \]

Therefore,
\[ \|x_{n+1} - x^*\|^2 \leq (1 - \xi_n)\|x_n - x^*\|^2 + \xi_n \varepsilon_n. \] (2.20)

Thanks to the condition (C1) and Eq.(2.20), we conclude that
\[ \lim_{n \to \infty} \xi_n = 0, \sum_{n=1}^{\infty} \xi_n = \infty. \]

From Lemma 1.10 we can obtain \( x_n \to x^*. \)

## 3 Numerical Example

In this section, we get one example is presented to grantee the main theorem (2.1).

**Example 3.1** Let \( H = \mathbb{R}, C = [-1, 1], G_1(x, y) = -3x^2 + xy + 2y^2, \)
\( G_2(x, y) = -4x^2 + xy + 3y^2 \) and \( G_3(x, y) = -9x^2 + xy + 8y^2. \) Also, we consider \( S_n = I, f(x) = \frac{x}{5} \) and \( A = I \) be a strongly positive linear bounded operator with coefficient \( \gamma = 1. \) It is easy to check that \( A \) and \( f \) satisfy all conditions in Theorem 2.2. For each \( r > 0 \) and \( x \in C, \) there exists \( z \in C \) such that, for any \( y \in C, \)
\[ G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \]
\[ \Leftrightarrow -3z^2 + zy + 2y^2 + \frac{1}{r} (y - z) (z - x) \geq 0 \]
\[ \Leftrightarrow 2ry^2 + ((r + 1)z - x)y - 3rz^2 - z^2 + zx \geq 0 \]

Set \( G(y) = 2ry^2 + ((r + 1)z - x)y - 3rz^2 - z^2 + zx. \) Then \( G(y) \) is a quadratic function of \( y \) with coefficients \( a = 2r, b = (r + 1)z - x \) and \( c = -3rz^2 - z^2 + zx. \) So
\[ \Delta = [(r + 1)z - x]^2 - 8r(zx - z^2 - 3rz^2) \]
\[ = (r + 1)^2z^2 - 2(r + 1)xz + x^2 + 24r^2z^2 + 8rz^2 - 8rxx \]
\[ = x^2 - 2(5rz + z)x + (25r^2z^2 + 10r^2z + z^2) \]
\[ = [(x - (5rz + z))]^2. \]

Since \( G(y) \geq 0 \) for all \( y \in C \), if and only if \( \Delta = [(x - (5rz + z))]^2 \leq 0. \)

Therefore, \( z = \frac{x}{5r + 1}, \) which yields \( T_{rn,1} = u_n^{(1)} = \frac{x}{5r + 1}. \)

By the same argument, for \( G_2 \) and \( G_3 \), one can conclude \( T_{rn,2} = u_n^{(2)} = \frac{x}{7r + 1}, T_{rn,3} = u_n^{(3)} = \frac{x}{17r + 1}. \) Let \( r_n = \frac{n}{n+1}. \) Hence

\[ \theta_n = \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)}}{3} = \frac{1280n^3 + 344n^2 + 67n + 3}{3864n^3 + 300n^2 + 32n + 1}. \]

Suppose that \( \alpha_n = \frac{2n - 1}{10n - 9}, \beta_n = \frac{1}{n} \) and \( \lambda_n = \epsilon, \) we have

\[ W_1 = U_{11} = \lambda_1 S_1 U_{12} + (1 - \lambda_1)I, \]
\[ W_2 = U_{21} = \lambda_1 S_1 U_{22} + (1 - \lambda_1)I \]
\[ = \lambda_1 S_1 \{\lambda_2 S_2 U_{23} + (1 - \lambda_2)I\} + (1 - \lambda_1)I, \]
\[ = \lambda_1 \lambda_2 S_1 S_2 + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \]
\[ W_3 = U_{31} = \lambda_1 S_1 U_{32} + (1 - \lambda_1)I \]
\[ = \lambda_1 S_1 \{\lambda_2 S_2 U_{33} + (1 - \lambda_2)I\} + (1 - \lambda_1)I, \]
\[ = \lambda_1 \lambda_2 S_1 S_2 U_{33} + \lambda_1 (1 - \lambda_2) S_1 + (1 - \lambda_1)I, \]
\[ = \lambda_1 \lambda_2 \lambda_3 S_1 S_2 S_3 + \lambda_2 (1 - \lambda_3) S_1 S_2 + \lambda_1 (1 - \lambda_2) S_2 + (1 - \lambda_1)I. \]

By iteration this manner, we have

\[ W_n = U_{n1} = \lambda_1 \lambda_2 \cdots \lambda_n S_1 S_2 \cdots S_n + \lambda_1 \lambda_2 \cdots \lambda_{n-1} (1 - \lambda_n) S_1 S_2 \cdots S_{n-1} \]
\[ + \lambda_1 \lambda_2 \cdots \lambda_{n-2} (1 - \lambda_{n-1}) S_1 S_2 \cdots S_{n-2} + \cdots + \lambda_1 (1 - \lambda_2) S_1 \]
\[ + (1 - \lambda_1)I. \]

Let \( T_n = I, \lambda_n = \epsilon, \) we obtain

\[ W_n = [\epsilon^n + \epsilon^{n-1}(1 - \epsilon) + \cdots + \epsilon(1 - \epsilon) + (1 - \epsilon)] I = I. \]
Hence
\[ y_n = \left( \frac{280n^3 + 344n^2 + 67n + 3}{864n^3 + 300n^2 + 32n + 1} \right) \left( \frac{15n - 14}{15n} \right)x_n. \]

We have the following algorithm for the sequence \( \{x_n\} \)

\[ x_{n+1} = \frac{2n - 1}{10n - 9} x_n + \frac{8n - 8}{10n - 9} y_n. \]

Choose \( x_1 = 1 \). By using MATLAB software, we obtain the following table and figure of the result.

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<th>( x_n )</th>
<th>( n )</th>
<th>( x_n )</th>
<th>( n )</th>
<th>( x_n )</th>
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</tr>
<tr>
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<td>0.001637456001</td>
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<td>0.0007885980277</td>
<td>23</td>
<td>0.0000004609899888</td>
</tr>
<tr>
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<td>0.0003784282172</td>
<td>24</td>
<td>0.0000002170787103</td>
</tr>
<tr>
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<tr>
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Fig. 1. The graph of \( \{x_n\} \) with initial value \( x_1 = 1 \).

References


