

# The Operational matrices with respect to generalized Laguerre polynomials and their applications in solving linear differential equations with variable coefficients

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Received 3 October 2011; accepted 22 July 2013

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## Abstract

In this paper, a new and efficient approach based on operational matrices with respect to the generalized Laguerre polynomials for numerical approximation of the linear ordinary differential equations (ODEs) with variable coefficients is introduced. Explicit formulae which express the generalized Laguerre expansion coefficients for the moments of the derivatives of any differentiable function in terms of the original expansion coefficients of the function itself are given in the matrix form. The main importance of this scheme is that using this approach reduces solving the linear differential equations to solve a system of linear algebraic equations, thus greatly simplify the problem. In addition, several numerical experiments are given to demonstrate the validity and applicability of the method.

*Key words:* Operational matrices, Laguerre polynomials, Linear differential equations with variable coefficients.

*2010 AMS Mathematics Subject Classification :* 33CXX, 34B15.

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## 1 Introduction

Orthogonal polynomials play a prominent role in pure, applied and computational mathematics, as well as in the applied sciences and also in many fields of numerical analysis such as quadratures, approximation theory and so on [1-4]. In particular case, these polynomials have an important role in the spectral methods. These methods (spectral methods) have been successfully applied in the approximation of partial, differential and integral equations. Three most widely used spectral versions are the Galerkin, collocation and Tau methods. Their utility is based on the fact that if the solution sought is smooth, usually only a few terms in an expansion of global basis functions are needed to represent it to high accuracy [5-11]. We must note to this point that numerical methods for ordinary, partial and integral differential equations can be classified into the local and global categories.

The finite-difference and finite-element methods are based on local arguments, whereas the spectral methods are in the global class [12,13]. Spectral methods, in the context of numerical schemes for differential equations, belong to the family of weighted residual methods, which are traditionally regarded as the foundation of many numerical methods such as finite element, spectral, finite volume and boundary element methods. Also the linear ODEs with variable coefficients and their solutions play a major role in the branch of modern mathematics and arise frequently in many applied areas. Therefore, a reliable and efficient technique for the solution of them is too important.

The analytic results on the existence and uniqueness of solutions to the second order linear ODEs have been investigated by many authors [14,15], however the existence and uniqueness of the solution for ODEs under their conditions is beyond the scope of this paper. We assume that the ODEs which we consider in this paper with their conditions have solutions. During the last decades, several methods have been used to solve higher order linear ODEs such as Adomian's decomposition method [16-18], Taylor collocation method [19-22], Haar functions method [23,24], Tau method [25,26], Wavelet method [27], Hybrid function method [28],

Legendre wavelet method [29], collocation method based on Jacobi polynomials [30], Taylor polynomial solutions [31], Boubaker polynomial approach [32], and Bernoulli polynomial approach [33]. In this paper, we develop a new and efficient approach to obtain the numerical solution of the general linear ODEs with variable coefficients of the form

$$\sum_{k=1}^{d_j} A_{k,j}(x)y^{(j)}(x) + \sum_{k=1}^{d_{j-1}} A_{k,j}(x)y^{(j-1)}(x) + \dots + \sum_{k=1}^{d_0} A_{k,j}(x)y^{(0)}(x) = g(x),$$

$$0 \leq x \leq \infty,$$

$$j \geq 0, d_t > 0, t = 0, \dots, j,$$
(1.1)

with the conditions

$$\sum_{k=0}^j \alpha_{ik} y^{(k)}(a_i) = \mu_i, \quad i = 0, 1, \dots, j.$$
(1.2)

The main importance of our work is considering the general linear ODEs (1.1) with respect to (1.2), wherein the other papers only considered particular cases of our general problem. Also using the generalized Laguerre polynomials as the basic functions for numerical approximation wherein the classical Laguerre polynomials are particular cases of them, is the other superiority of our paper. The remainder of this paper is organized as follows: In section 2, we introduce the properties of generalized Laguerre polynomials and the basic formulation of them required for our subsequent development. Section 3, is devoted to the operational matrices of generalized Laguerre polynomials (derivative and moment) with some useful theorems. Section 4, summarizes the application of generalized Laguerre polynomials to the solution of problem (1.1) and (1.2). Thus, a set of linear equations is formed and a solution of the considered problem is introduced. Section 5, is devoted to approximations by generalized Laguerre polynomials and a useful theorem. In section 6, the proposed method is applied for three numerical experiments. An application of the method for the high order linear differential equation is presented in section 7. Finally, we have monitored a brief conclusion in section 8. Note that we have computed the numerical results by Matlab (version 2013) programming.

## 2 The generalized Laguerre polynomials

In this part, we define the generalized Laguerre polynomials and their properties such as their Sturm-Liouville ODE, three terms recursion formula and etc. Let  $\Lambda = (0, +\infty)$ , then Laguerre polynomials are denoted by  $L_n^\alpha(x)$  ( $\alpha > -1$ ), and they are the eigenfunctions of the Sturm-Liouville problem

$$x^{-\alpha} e^x (x^{\alpha+1} e^{-x} (L_n^\alpha(x))')' + \lambda_n L_n^\alpha(x) = 0, \quad x \in \Lambda, \quad (2.1)$$

with the eigenvalues  $\lambda_n = n$  [13].

Laguerre polynomials are orthogonal in  $L_{w_\alpha}^2(\Lambda)$  space with the weight function  $w_\alpha(x) = x^\alpha e^{-x}$ , satisfy in the following relation

$$\int_0^{+\infty} L_n^\alpha(x) L_m^\alpha(x) w_\alpha(x) dx = \gamma_n^\alpha \delta_{m,n}, \quad \gamma_n^\alpha = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)}, \quad (2.2)$$

where  $\delta_{m,n}$  is a kronecker delta function. The explicit form of these polynomials is in the form

$$L_n^\alpha(x) = \sum_{i=0}^n E_i^\alpha x^i, \quad (2.3)$$

where

$$E_i^\alpha = \frac{\binom{n + \alpha}{n - i} (-1)^i}{i!}. \quad (2.4)$$

These polynomials are satisfied in the following three terms recurrence formula

$$\begin{aligned} (n + 1)L_{n+1}^\alpha(x) &= (2n + \alpha + 1 - x)L_n^\alpha(x) - (n + \alpha)L_{n-1}^\alpha(x), \\ L_0^\alpha(x) &= 1, L_1^\alpha(x) = \alpha + 1 - x. \end{aligned} \quad (2.5)$$

The case  $\alpha = 0$  leads to the classical Laguerre polynomials, which are used most frequently in practice and will simply be denoted by  $L_n(x)$ . An important property of the Laguerre polynomials is the following derivative relation [13]:

$$(L_n^\alpha(x))' = \sum_{i=0}^{n-1} L_i^\alpha(x). \quad (2.6)$$

Further,  $(L_i^\alpha(x))^{(k)}$  are orthogonal with respect to the weight function  $w_{\alpha+k}$ . i.e.,

$$\int_0^{+\infty} (L_i^\alpha)^{(k)}(x)(L_j^\alpha)^{(k)}(x)w_{\alpha+k}(x)dx = \gamma_{n-k}^{\alpha+k}\delta_{i,j}, \quad (2.7)$$

where  $\gamma_{n-k}^{\alpha+k}$  is defined in (2.7).

A function  $y(x) \in L_{w_\alpha}^2[0, \infty)$ , can be expressed in terms of the generalized Laguerre polynomials as

$$y(x) = \sum_{i=0}^{\infty} a_i L_i^\alpha(x), \quad (2.8)$$

where the coefficients  $a_i$  are given by

$$a_i = \frac{1}{\gamma_i^\alpha} \int_0^{+\infty} L_i^\alpha(x)y(x)w^{(\alpha)}(x)dx. \quad (2.9)$$

In practice, only the first  $m + 1$  terms of the generalized Laguerre polynomials are considered. Then we have

$$y_m(x) = \sum_{i=0}^m a_i L_i^\alpha(x) = L^\alpha(x)^T A, \quad (2.10)$$

where the generalized Laguerre polynomials coefficients vector  $A$  and the generalized Laguerre polynomials vector  $L^{(\alpha)}(x)$  are given by

$$A = [a_0, a_1, \dots, a_m]^T, \quad (2.11)$$

$$L^{(\alpha)}(x) = [L_0^\alpha(x), L_1^\alpha(x), \dots, L_m^\alpha(x)]^T.$$

### 3 The operational matrices of the Laguerre polynomials (derivative and moment)

In this section, we present the operational matrices of generalized Laguerre polynomials (derivative and moment). To do this, first we introduce the concept of the operational matrix.

#### 3.1 The operational matrix

**Definition 1.** Suppose

$$\phi = [\phi_0, \phi_1, \dots, \phi_n], \quad (3.1)$$

where  $\phi_0, \phi_1, \dots, \phi_n$  are the basic functions on the given interval  $[a, b]$ . The matrices  $E_{n \times n}$  and  $F_{n \times n}$  are named as the operational matrices of derivatives and integrals respectively if and only if

$$\begin{aligned} \frac{d}{dt} \phi(t) &= E \phi(t), \\ \int_a^x \phi(t) dt &\simeq F \phi(t). \end{aligned} \quad (3.2)$$

Further assume  $g = [g_0, g_1, \dots, g_n]$ , named as the operational matrix of the product, if and only if

$$\phi(x) \phi^T(x) \simeq G_g \phi(x). \quad (3.3)$$

In other words, to obtain the operational matrix of a product, it is sufficient to find  $g_{i,j,k}$  in the following relation

$$\phi_i(x) \phi_j(x) \simeq \sum_{k=0}^{i+j} g_{i,j,k} \phi_k(x), \quad (3.4)$$

which is called the linearization formula [34]. Operational matrices are used in several areas of numerical analysis and they hold particular importance in various subjects such as integral equations [35], differential and partial differential equations [36] and etc. Also many textbooks and

papers have employed the operational matrices for spectral methods. Now we present the following theorem.

**Theorem 1.** If we consider the generalized Laguerre approximation

$$y(x) \cong \sum_{i=0}^m a_i L_i^{(\alpha)}(x) = \left( L^{(\alpha)}(x) \right)^T A, \quad (3.5)$$

then

$$x^i y^{(j)}(x) \cong B^T L^{(\alpha)}(x) = \left( (G^i D^j)^T A \right)^T L^{(\alpha)}(x), \quad (3.6)$$

where

$$D_{i,j} = \begin{cases} 1, & i > j, \\ 0, & i \leq j. \end{cases} \quad (3.7)$$

and

$$G_{i,j} = \begin{cases} -(i + \alpha), & j = i - 1, \\ -i, & j = i, \\ -(i + \alpha), & j = i + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3.8)$$

**Proof:** First, we obtain the operational matrix with respect to the derivative operator. For this goal, we must obtain a matrix  $D$  which satisfy in the following formula

$$\begin{bmatrix} \left( L_0^{(\alpha)}(x) \right)' \\ \left( L_1^{(\alpha)}(x) \right)' \\ \vdots \\ \left( L_n^{(\alpha)}(x) \right)' \end{bmatrix} = D \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}, \quad (3.9)$$

but by using (2.6), we can obtain the matrix  $D$  as the following

$$D_{i,j} = \begin{cases} 1, & i > j, \\ 0, & i \leq j. \end{cases} \quad (3.10)$$

Now by  $j$ -times repeating the formula (3.9), we can obtain the operational matrix with respect to  $y^{(j)}(x)$  as the following

$$\begin{bmatrix} (L_0^{(\alpha)}(x))^j \\ (L_1^{(\alpha)}(x))^j \\ \vdots \\ (L_n^{(\alpha)}(x))^j \end{bmatrix} = D^j \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}. \quad (3.11)$$

Also for obtaining the operational matrix with respect to the moment operator we must obtain a matrix  $G$ , which satisfy in the following relation

$$\begin{bmatrix} xL_0^{(\alpha)}(x) \\ xL_1^{(\alpha)}(x) \\ \vdots \\ xL_n^{(\alpha)}(x) \end{bmatrix} = G \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}, \quad (3.12)$$

but by using (2.5), we can obtain the matrix  $G$  as the following

$$G_{i,j} = \begin{cases} -(i + \alpha), & j = i - 1, \\ -i, & j = i, \\ -(i + \alpha), & j = i + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3.13)$$

Now by  $j$ -times repeating the formula (3.12), we can obtain the operational matrix with respect to  $x^j y(x)$ , as the following

$$\begin{bmatrix} x^j L_0^{(\alpha)}(x) \\ x^j L_1^{(\alpha)}(x) \\ \vdots \\ x^j L_n^{(\alpha)}(x) \end{bmatrix} = G^j \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}. \quad (3.14)$$



Now using formulae (3.11) and (3.14), yields

$$x^i y^{(j)}(x) \simeq \sum_{k=0}^n a_k x^i (L_k^{(\alpha)}(x))^{(j)} = A^T x^i \begin{bmatrix} (L_0^{(\alpha)}(x))^{(j)} \\ (L_1^{(\alpha)}(x))^{(j)} \\ \vdots \\ (L_n^{(\alpha)}(x))^{(j)} \end{bmatrix} = \quad (3.15)$$

$$A^T G^i D^j \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix} = ((G^i D^j)^T A)^T L^{(\alpha)}(x),$$

so the proof is completed.  $\square$

#### 4 The method of solution

In this section, we describe our new approach for solving the linear differential equations with variable coefficients (1.1), with respect to the conditions (1.2). Our approach is based on approximating the exact solution of (1.1), by truncating the generalized Laguerre expansion as

$$y(x) \simeq \sum_{i=0}^m a_i L_i^{(\alpha)}(x) = (L^{(\alpha)}(x))^T A, \quad (4.1)$$

where

$$A = [a_0, a_1, \dots, a_m]^T, \quad (4.2)$$

and

$$L^{(\alpha)}(x) = [L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \dots, L_m^{(\alpha)}(x)]^T. \quad (4.3)$$

Also we assume that the coefficients  $A_{k,j}(x)$  have the Taylor series expansion in the following form

$$A_{k,j}(x) = \sum_{i=0}^{m_j} e_{k,i}^{(j)} x^i. \quad (4.4)$$

Now by substituting Eqs. (4.1) and (4.4) into Eq. (1.1), we obtain

$$\sum_{k=1}^{s_j} \sum_{i=0}^{m_j} e_{k,i}^{(j)} x^i y^{(j)}(x) + \sum_{k=1}^{s_{j-1}} \sum_{i=0}^{m_{j-1}} e_{k,i}^{(j-1)} x^i y^{(j-1)}(x) + \dots + \sum_{k=1}^{s_0} \sum_{i=0}^{m_0} e_{k,i}^{(0)} x^i y^{(j)}(x) \simeq f(x), \quad (4.5)$$

therefore from Eq. (4.5), we must simplify  $x^i (y^{(j)}(x))$  as the following

$$x^i y^{(j)}(x) \simeq \sum_{i=0}^m a_i L_i^{(\alpha)}(x) = (L^{(\alpha)}(x))^T B_{(j)}^{(i)} = (G^i D^j)^T A^T (L^{(\alpha)}(x)), \quad (4.6)$$

where  $D$  and  $G$ , are defined in Eqs. (3.7) and (3.8), respectively. Also we approximate the right hand side of Eq. (1.1), as

$$f(x) = \sum_{i=0}^m b_i L_i^{(\alpha)}(x) = (L^{(\alpha)}(x))^T B, \quad (4.7)$$

where

$$B = [b_0, b_1, \dots, b_m]^T, \quad (4.8)$$

and

$$L^{(\alpha)}(x) = [L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \dots, L_m^{(\alpha)}(x)]^T. \quad (4.9)$$

Using Eqs. (4.6) and (4.7), into Eq. (4.5), we obtain

$$(L^{(\alpha)}(x))^T \left( \sum_{k=1}^{s_j} \sum_{i=0}^{m_j} e_{i,k}^{(j)} B_{(j)}^{(i)} + \sum_{k=1}^{s_{j-1}} \sum_{i=0}^{m_{j-1}} e_{i,k}^{(j-1)} B_{(j-1)}^{(i)} + \dots + \sum_{k=1}^{s_0} \sum_{i=0}^{m_0} e_{i,k}^{(0)} B_{(0)}^{(i)} \right) = (L^{(\alpha)}(x))^T F \simeq (L^{(\alpha)}(x))^T B. \quad (4.10)$$

From linear independency of the generalized Laguerre polynomials, we conclude

$$F = B, \quad (4.11)$$

where

$$F = [f_0, f_1, \dots, f_m]. \quad (4.12)$$

Therefore from identity (4.11), we have a system of  $m + 1$  algebraic equations of  $m + 1$  unknown coefficients  $a_i (i = 0, \dots, m)$ . Finally, we must obtain the corresponding matrix form of the boundary conditions. For

this purpose from Eq. (1.2), the values  $y^{(j)}(a)$  can be written as

$$y^{(j)}(a) = \left(L^{(\alpha)}(a)\right)^T \left(D^j\right)^T A, \quad a \in [0, +\infty). \quad (4.13)$$

Substituting (4.13), in the boundary conditions (1.2) and then simplifying it, we obtain the following matrix form

$$\sum_{i=0}^j b_{i,l} y^{(l)}(a_i) = \left(L^{(\alpha)}(a_i)\right)^T \left\{ \sum_{i=0}^j b_{i,l} D^i A \right\} = \sigma_l, \quad a_i \in [0, +\infty). \quad (4.14)$$

Now from Eqs. (4.11) and (4.14), we have  $m+j+1$  algebraic equations of  $m+1$  unknown coefficients. Thus for obtaining the unknown coefficients, we must eliminate  $j$  arbitrary equations from these  $m+j+1$  equations. But because of the necessity of holding the boundary conditions, we eliminate the last  $j$  equations from (4.11). Finally, replacing the last  $j$  equations of (4.11) by the  $j$  equations of (4.14), we obtain a system of  $m+1$  equations of  $m+1$  unknowns  $a_i (i = 0, \dots, m)$ .

## 5 Approximations by Generalized Laguerre polynomials

Now in this section, we present some useful theorems which show the approximations of functions by the generalized Laguerre polynomials. For this purpose, let us define  $\Lambda = \{x \mid 0 \leq x < \infty\}$  and

$$J_N^{(\alpha)} = \text{span}\{L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \dots, L_n^{(\alpha)}(x)\}.$$

The  $L_{w(\alpha)}^2(\Lambda)$ -orthogonal projection  $\pi_N^{(\alpha)} : L^2(\Lambda) \rightarrow J_N^{(\alpha)}$  is a mapping in a way that for any  $y(x) \in L^2(\Lambda)$ , we have

$$\langle \pi_N^{(\alpha)}(y) - y, \Phi \rangle = 0, \quad \forall \Phi \in J_N^{(\alpha)}.$$

Due to the orthogonality, we can write

$$\pi_N^{(\alpha)}(y) = \sum_{k=0}^{N-1} c_k L_k^{(\alpha)}(x), \quad (5.1)$$

where  $c_i$  ( $i = 0, 1, \dots, N - 1$ ) are constants in the following form

$$c_i = \frac{1}{\gamma_k^{(\alpha)}} \langle y(x), L_k^{(\alpha)} \rangle_{L^2_{w^{(\alpha)}}} .$$

In the literature of spectral methods,  $\pi_N^{(\alpha)}(y)$  is named as the generalized Laguerre expansion of  $y(x)$  and approximates  $y(x)$  on  $(0, +\infty)$ . In the spectral methods, by substituting the generalized Laguerre expansion  $\pi_N^{(\alpha)}(y)$  in the ODEs and their boundary conditions, we obtain a residual term which is symbolically showed by  $Res(x)$  as a function of  $x, N$ , and  $\alpha$ . Different strategies for minimizing a residual term  $Res(x)$ , lead to the different versions of spectral methods such as Galerkin, Tau and collocation methods. For instance, in the collocation methods the residual term  $Res(x)$  is vanished in particular points named as collocated points. Also estimating the distance between  $y(x)$  and its generalized Laguerre expansion as measured in the weighted norm  $\|\cdot\|_{w_\alpha}$  is an important problem in numerical analysis. The following theorem provides the basic approximation results for generalized Laguerre expansion.

**Theorem 2.** we have

$$\left\| \frac{d^l}{dx^l} (\pi_N^{(\alpha)}(y) - y) \right\|_{w^{(\alpha+l)}} \leq N^{(l-m)/2} \left\| \frac{d^m}{dx^m} y(x) \right\|_{w_{\alpha+m}},$$

$$0 \leq l \leq m, \quad \forall y \in B_{(\alpha)}^m(\Lambda),$$

where

$$B_{(\alpha)}^m(\Lambda) = \left\{ \forall y \in L^2_{w_\alpha} : \frac{d^l y}{dx^l} \in L^2_{w_{\alpha+l}}(\Lambda), \quad 0 \leq l \leq m \right\}.$$

**Proof:** See [13].  $\square$

## 6 The test experiments

In this section, some numerical experiments are given to illustrate the properties of the method and all of them were performed on a computer using a program written in Matlab 2013.

**Experiment 1.** Consider the following linear differential equation with variable coefficients [37],

$$y''(x) + xy'(x) + xy(x) = 1 + x + x^2, \quad -1 \leq x \leq 1, \quad (6.1)$$

$$y(0) = 1, \quad y'(0) + 2y(1) - y(-1) = -1.$$

Now we approximate the exact solution of (6.1), by

$$y(x) \simeq \sum_{i=0}^6 a_i L_i^{(\alpha)}(x) = \left(L^{(\alpha)}(x)\right)^T A, \quad (6.2)$$

where

$$A = [a_0, a_1, \dots, a_6]. \quad (6.3)$$

Also we expand the right hand side of (6.1) as

$$1 + x + x^2 = \sum_{i=0}^6 b_i L_i^{(\alpha)}(x) = \left(L^{(\alpha)}(x)\right)^T B, \quad (6.4)$$

where

$$B = [1, 2, 3/2, 0, 0, 0, 0]. \quad (6.5)$$

First we reduce Eq. (6.1) into the following matrix form

$$(D^2 + GD + G^T)A = B. \quad (6.6)$$

Also its boundary conditions as

$$\sum_{i=0}^6 a_i L_i^{(\alpha)}(0) = \left(L^{(\alpha)}(0)\right)^T A = 0. \quad (6.7)$$

and

$$\sum_{i=0}^6 a_i L_i^{(\alpha)}(1) = (L^{(\alpha)}(1))^T A = 1. \quad (6.8)$$

By implementation of our method which is presented in section 4, and also after the augmented matrices of the system and boundary conditions are computed, we obtain the numerical solutions. The comparison between our method and Taylor method is shown in table 1. Also the approximate and exact solutions are shown in figure 1.

Table 1. The comparison between our method ( $n = 6$  and  $\alpha = 3$ ) and the approximate solutions of Taylor method ( $n = 4$ ) of experiment 1.

x	Our method	Taylor method
-1.0	-1.0000000000	-0.9999999999
-0.8	-0.8000000000	-0.7999999996
-0.6	-0.6000000000	-0.5999999998
-0.4	-0.3879999999	-0.4000000000
-0.2	-0.1889999999	-0.2000000000
0.0	-0.9999999999	-1.0000000000
0.2	0.1999999999	0.2000000000
0.4	0.3899999999	0.4000000000
0.6	0.6000000000	0.5999999999
0.8	0.8000000000	0.7999999997
1.0	1.0000000000	0.9999999988

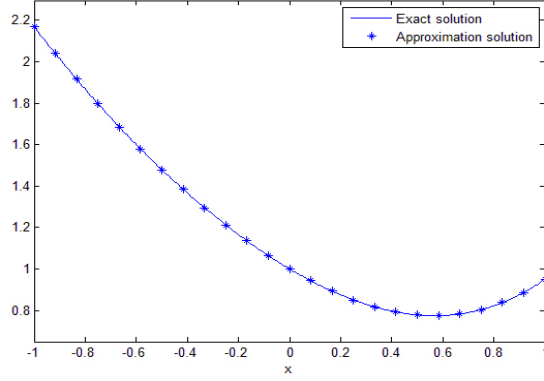


Figure 1. The comparison between exact and approximate solutions of  $n = 14$  and  $\alpha = 3/2$  of experiment 1.

**Experiment 2.** Consider the second-order differential equation

$$(x^2 + 1)y''(x) + y'(x) = 1, \quad (6.9)$$

with the boundary conditions

$$y(0) = 0, y(1) = 1. \quad (6.10)$$

The exact solution of (6.9) is  $y(x) = x$ .

Now we approximate the exact solution of (6.9), by

$$y(x) \simeq \sum_{i=0}^5 a_i L_i^{(\alpha)}(x) = (L^{(\alpha)}(x))^T A, \quad (6.11)$$

where

$$A = [a_0, a_1, \dots, a_5]. \quad (6.12)$$

Also we expand the right hand side of (6.9) as

$$1 \simeq \sum_{i=0}^5 b_i L_i^{(\alpha)}(x) = (L^{(\alpha)}(x))^T B, \quad (6.13)$$

where

$$B = [1, 0, 0, 0, 0, 0]. \quad (6.14)$$

Now, first we reduce Eq. (6.9) into the following matrix form

$$(G^2 D^2 + D^2 + D)^T A = B. \quad (6.15)$$

Also its boundary conditions as

$$\sum_{i=0}^5 a_i L_i^{(\alpha)}(0) = (L^{(\alpha)}(0))^T A = 0. \quad (6.16)$$

and

$$\sum_{i=0}^5 a_i L_i^{(\alpha)}(1) = (L^{(\alpha)}(1))^T A = 1. \quad (6.17)$$

By implementation of our method which is presented in section 4, and also after the augmented matrices of the system and boundary conditions are computed, we obtain the solution

$$y(x) = x, \quad (6.18)$$

which is the exact solution.

**Experiment 3.** Consider the third-order linear differential equation

$$x^2 y'''(x) + y''(x) = 2, \quad (6.19)$$

$$y(0) = 0, y(1) = 1, y(-1) = 1. \quad (6.20)$$

Now we approximate the exact solution of (6.19) by

$$y(x) \simeq \sum_{i=0}^5 a_i L_i^{(\alpha)}(x) = (L^{(\alpha)}(x))^T A. \quad (6.21)$$

Also we expand the right hand side of (6.19) as

$$2 \simeq \sum_{i=0}^5 b_i L_i^{(\alpha)}(x) = (L^{(\alpha)}(x))^T B, \quad (6.22)$$

where

$$B = [2, 0, 0, 0, 0]. \quad (6.23)$$

Now we must reduce Eq. (6.19) into the following matrix form

$$(G^2 D^3 + D^2)^T A = B. \quad (6.24)$$



and also its boundary conditions as

$$\sum_{i=0}^5 a_i L_i^{(\alpha)}(0) = \left( L^{(\alpha)}(0) \right)^T A = 0, \quad (6.25)$$

$$\sum_{i=0}^5 a_i L_i^{(\alpha)}(1) = \left( L^{(\alpha)}(1) \right)^T A = 1. \quad (6.26)$$

and

$$\sum_{i=0}^5 a_i L_i^{(\alpha)}(-1) = \left( L^{(\alpha)}(1) \right)^T A = 1. \quad (6.27)$$

After the augmented matrices of the system and boundary conditions are computed, we obtain the solution

$$y(x) = x^2, \quad (6.28)$$

which is the exact solution.

## 7 Application of the method for the high order linear differential equation

In this section, we report the numerical results obtained for a high order linear differential equation by the above mentioned procedure. This shows that it is straightforward to extend the method to the high order linear differential equations as follows.

**Experiment 4.** Let us consider the eighth order linear differential equation [38,39]

$$y^{(8)}(x) - y(x) = -8e^x, \quad 0 \leq x \leq 1, \quad (7.1)$$

with the initial conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1,$$

$$y'''(0) = -2, \quad y^{(4)}(0) = -3, \quad y^{(5)}(0) = -4, \quad (7.2)$$

$$y^{(6)}(0) = -5, \quad y^{(7)}(0) = -6.$$

The exact solution of this equation is  $y(x) = (1-x)e^x$ . By implementation of our method which is presented in section 4, and also after the augmented matrices of the system and boundary conditions are computed, we obtain the numerical solutions. The comparison between our method and other numerical methods are shown in tables 2 and 3. Also the exact and approximate solutions are shown in figure 2. We see that our method, HPM and MDM methods obtain good results than the other methods for this experiment. These methods rather than the Taylor polynomial set obtain better results near the corner of interval. In other words, in the interior points between 0 and 1, the Taylor method give better results. This matter is seen by [17,18] also, which is due to the affinity of Taylor series to the origin.

Table 2. The comparison between the exact and approximate solutions of HPM, MDM and present methods of experiment 4.

x	Exact	Present method	HPM method	MDM method
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9946538262	0.9946538261	0.9946538263	0.9946538262
0.2	0.9771222065	0.9771222063	0.9771222065	0.9771222014
0.3	0.9449011653	0.9449011651	0.9449011653	0.9449010769
0.4	0.8950948185	0.8950948183	0.8950948186	0.8950941522
0.5	0.8243606353	0.8243606351	0.8243606356	0.8243574386
0.6	0.7288475201	0.7288475205	0.728847522	0.7288359969
0.7	0.6041258122	0.6041258121	0.6041258211	0.6040917111
0.8	0.4451081856	0.4451081852	0.445108220	0.4450208387
0.9	0.2459603111	0.2459603101	0.2459604249	0.2457599482
1.0	0.0000000000	0.0000000000	$3.326 \times 10^{-7}$	$4.212943 \times 10^{-4}$

Table 3. The comparison between the exact and approximate solutions of present and Taylor methods of experiment 4.

x	Exact	Present method	Taylor method
0.0	1.0000000000	1.0000000000	1.0000000000
0.1	0.9946538262	0.9946538261	0.9946538266
0.2	0.9771222065	0.9771222063	0.9771222093
0.3	0.9449011653	0.9449011651	0.9449011752
0.4	0.8950948185	0.8950948183	0.8950948487
0.5	0.8243606353	0.8243606351	0.8243607328
0.6	0.7288475201	0.7288475205	0.7288478604
0.7	0.6041258122	0.6041258121	0.6041269662
0.8	0.4451081856	0.4451081852	0.4451117669
0.9	0.2459603111	0.2459603101	0.2459703618
1.0	0.0000000000	0.0000000000	$2.57 \times 10^{-5}$

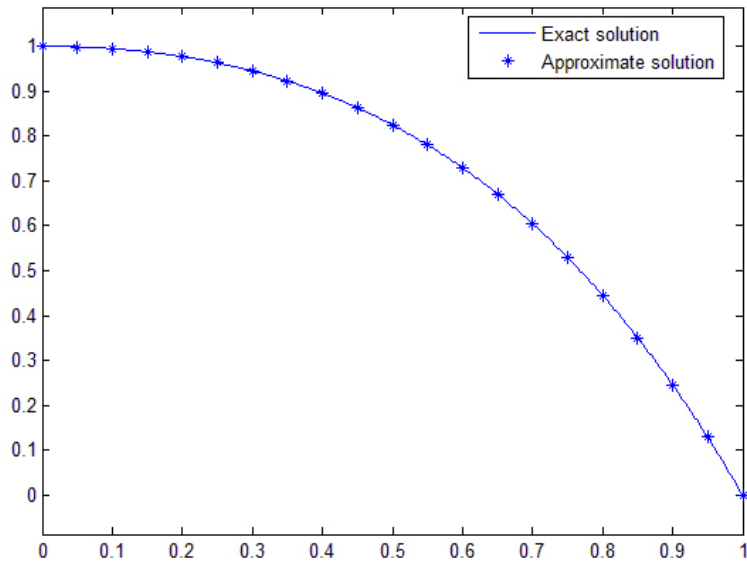


Figure 2. The comparison between the exact and approximate solutions of  $n = 12$  and  $\alpha = 2$  of experiment 4.

## 8 Conclusion

In this paper, we have introduced a new and efficient approach for numerical approximation of the linear differential equations with variable coefficients. The method is based on the approximation of the exact solution with the generalized Laguerre polynomials approximation and also variable coefficients with Taylor series expansion. Implementation of the method reduces the problem to a system of algebraic equations. In addition, application of the method for numerical solution of high order ODEs is presented. Finally, some test experiments are presented for showing the accuracy and efficiency of the present method with the other methods such as HPM, MDM and Taylor series.

## Acknowledgement

We are grateful to the reviewers and the editor Professor Dr. Saeid Abbasbandy for their helpful comments and suggestions which indeed improved the quality of this manuscript.

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