



Generalized composition operators from logarithmic Bloch type spaces to \mathcal{Q}_K type spaces

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Abstract

In this paper boundedness and compactness of generalized composition operators from logarithmic Bloch type spaces to \mathcal{Q}_K type spaces are investigated.

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1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk \mathbb{D} in the complex plane \mathbb{C} and $\alpha \in (0, \infty)$. The Bloch type space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$b_\alpha(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The little Bloch type space \mathcal{B}_0^α consists of those functions $f \in \mathcal{B}^\alpha$ for which

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

For $\beta \in [0, \infty)$, the logarithmic Bloch type space $\mathcal{B}_{\log^\beta}^\alpha = \mathcal{B}_{\log^\beta}^\alpha(\mathbb{D})$ introduced by Stevic in [9], is the space of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$b_{\alpha, \beta}(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2} \right)^\beta |f'(z)| < \infty.$$

The little logarithmic Bloch type space $\mathcal{B}_{\log^\beta, 0}^\alpha$ consists of those functions $f \in \mathcal{B}_{\log^\beta}^\alpha$ for which

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2} \right)^\beta |f'(z)| = 0.$$

In some papers (see [9]), the definitions of this kind of spaces are based on the coefficient $1 - |z|$, instead of $1 - |z|^2$. We first show that these are equivalent.

Obviously, $1 - |z| < 1 - |z|^2 < 2(1 - |z|)$ and $\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2} < \ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|}$ for all $z \in \mathbb{D}$. On the other hand,

$$\begin{aligned} \ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|} &= \ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2} + \ln(1 + |z|) \\ &< \ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2} + \ln \frac{e}{1 - |z|^2} \\ &= \left(1 + \frac{\alpha}{\beta}\right) \ln e^{\frac{\beta}{\alpha}} + 2 \ln \frac{1}{1 - |z|^2} \\ &\leq \max\left\{1 + \frac{\alpha}{\beta}, 2\right\} \ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2}. \end{aligned}$$

Therefore, we can replace $1 - |z|^2$ by $1 - |z|$ in the definitions of Bloch type spaces and logarithmic Bloch type spaces.

The space $\mathcal{B}_{\log^\beta}^\alpha$ is a Banach space with the norm $\|f\| := |f(0)| + b_{\alpha,\beta}(f)$, and $\mathcal{B}_{\log^\beta,0}^\alpha$ is a closed subspace of $\mathcal{B}_{\log^\beta}^\alpha$. If $\beta = 0$, then $\mathcal{B}_{\log^\beta}^\alpha$ ($\mathcal{B}_{\log^\beta,0}^\alpha$) coincides with the Bloch type space \mathcal{B}^α (little Bloch type space \mathcal{B}_0^α). For $\beta = 1$, the space $\mathcal{B}_{\log^\beta}^\alpha$ is the generally weighted Bloch space (see [5]). When $\alpha = \beta = 1$, the space $\mathcal{B}_{\log^\beta}^\alpha$ is just the weighted Bloch space \mathcal{B}_{\log} .

For $p \in (0, \infty)$ and $\alpha > -1$, the weighted Bergman space \mathcal{A}_α^p is the space of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{A}_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized area measure on \mathbb{D} . It is well known that \mathcal{A}_α^p is a Banach space for $p \geq 1$, and in the case that $0 < p < 1$, it is a complete metric space with the distance $d(f, g) = \|f - g\|_{\mathcal{A}_\alpha^p}^p$. In the special case when $p = 2$, \mathcal{A}_α^2 is a Hilbert space. For a general background about weighted Bergman spaces we refer to [16].

For $p \in (0, \infty)$ and $\alpha > -1$, the weighted Dirichlet type space \mathcal{D}_α^p is the space of all $f \in \mathcal{H}(\mathbb{D})$ for which $f' \in \mathcal{A}_\alpha^p$. Note that \mathcal{D}_α^p is a Banach space with the norm $\|f\| := |f(0)| + \|f'\|_{\mathcal{A}_\alpha^p}$. When $\alpha = 0$, \mathcal{D}_α^p coincides with the Dirichlet space \mathcal{D}^p .

For $a \in \mathbb{D}$, $G(z, a) = \log \frac{1}{|\sigma_a(z)|}$ is the Green's function on \mathbb{D} , where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of \mathbb{D} . For $s \in (0, \infty)$, the space \mathcal{Q}_s consists of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 G^s(z, a) dA(z) < \infty,$$

and its closed subspace $\mathcal{Q}_{s,0}$ consists of those functions $f \in \mathcal{Q}_s$ such that

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(z)|^2 G^s(z, a) dA(z) = 0.$$

It is well known that $\mathcal{Q}_1 = BMOA$ ($\mathcal{Q}_{1,0} = VMOA$), the space of all analytic functions of bounded (vanishing) mean oscillation [1].

In [15], Zhao introduced a general family of analytic function spaces, called the $F(p, q, s)$ -spaces with $p \in (1, \infty)$, $q \in (-2, \infty)$ and $s \in [0, \infty)$, consisting of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q G^s(z, a) dA(z) < \infty.$$

The closed subspace $F_0(p, q, s)$ of $F(p, q, s)$ consists of those functions $f \in F(p, q, s)$ such that

$$\lim_{|a| \rightarrow 1} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q G^s(z, a) dA(z) = 0.$$

If $q + s \leq -1$, $F(p, q, s)$ reduces to the space of constant functions. The interest in the $F(p, q, s)$ -spaces arises from the fact that they cover a lot of well-known function spaces which are listed in the following.

- $F(2, 0, s) = \mathcal{Q}_s$, $F_0(2, 0, s) = \mathcal{Q}_{s,0}$
- $F(2, 0, s) = \mathcal{B}$, $F_0(2, 0, s) = \mathcal{B}_0$ ($s > 1$)
- $F(2, 0, 1) = BMOA$, $F_0(2, 0, 1) = VMOA$
- $F(p, pq - 2, s) = \mathcal{B}^q$, $F_0(p, pq - 2, s) = \mathcal{B}_0^q$ ($s > 1$)
- $F(p, pq - 2, 1) = BMOA_p^q$ (The BMOA type spaces)
- $F_0(p, pq - 2, 1) = VMOA_p^q$ (The VMOA type spaces)
- $F(p, q, 0) = \mathcal{A}_{q-p}^p$ ($q - p > -1$), $F(p, q, 0) = \mathcal{D}_q^p$ ($q > -1$)
- $F(2, 1, 0) = H^2$ (The Hardy space)

In [10] Wulan and Zhou introduced a new space, \mathcal{Q}_K type space. For a right-continuous and nondecreasing function $K : [0, \infty) \rightarrow [0, \infty)$, and for $p \in (0, \infty)$, $q \in (-2, \infty)$, the \mathcal{Q}_K type space denoted by $\mathcal{Q}_K(p, q)$ consists of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{K,p,q}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) < \infty.$$

The space $\mathcal{Q}_K(p, q)$ is a Banach space with the norm $\|f\|_{\mathcal{Q}_K(p,q)} := |f(0)| + \|f\|_{K,p,q}$, when $p \geq 1$. The closed subspace $\mathcal{Q}_{K,0}(p, q)$ of $\mathcal{Q}_K(p, q)$ consists of those functions $f \in \mathcal{Q}_K(p, q)$ such that

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) = 0.$$

If $q + 2 = p$, $\mathcal{Q}_K(p, q)$ is Möbius invariant i.e, $\|f \circ \sigma_a\|_{K,p,q} = \|f\|_{K,p,q}$ for all $a \in \mathbb{D}$. We say that the space $\mathcal{Q}_K(p, q)$ is trivial if it contains constant functions only. For example, if $\int_0^1 (1 - r^2)^q K(\log \frac{1}{r}) r dr = \infty$, then $\mathcal{Q}_K(p, q)$ is trivial [10]. Also by [10, Theorem 3.1], if $K(1) > 0$ then the kernel function K can be chosen as bounded. Throughout the paper, we assume $K(1) > 0$ and

$$\int_0^1 (1 - r^2)^q K(\log \frac{1}{r}) r dr < \infty.$$

By [10, Theorem 2.1] we have $\mathcal{Q}_K(p, q) \subseteq \mathcal{B}^{\frac{q+2}{p}}$ and for a fixed $r \in (0, 1)$,

$$\|f\|_{K,p,q}^p \geq \pi r^2 K(\log \frac{1}{r}) b_{\frac{q+2}{p}}^p(f),$$

for all $f \in \mathcal{Q}_K(p, q)$. In the sequel, we use the inequality

$$b_{\frac{q+2}{p}}(f) \leq C \|f\|_{K,p,q}. \quad (1.1)$$

Also by [10, Theorem 2.1], we have $\mathcal{Q}_K(p, q) = \mathcal{B}^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 (1 - r^2)^{-2} K(\log \frac{1}{r}) r dr < \infty.$$

Now we recall some particular cases. If $p = 2$, $q = 0$, we have that $\mathcal{Q}_K(p, q) = \mathcal{Q}_K$. For more details on the spaces of \mathcal{Q} classes we refer to [4, ?, 12]. For $s \in [0, \infty)$, if $K(t) = t^s$, then $\mathcal{Q}_K(p, q) = F(p, q, s)$.

Let φ be an analytic self-map of \mathbb{D} and $g \in \mathcal{H}(\mathbb{D})$, the generalized composition operator C_φ^g is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi)) g(\xi) d\xi, \quad f \in \mathcal{H}(\mathbb{D}), \quad z \in \mathbb{D},$$

which is introduced in [6]. When $g = \varphi'$, this operator is essentially (up to a constant) the composition operator C_φ , which is defined by $C_\varphi f = f \circ \varphi$. Darus and Ibrahim has defined an integral operator on a class of analytic functions in the unit disk [3]. Zhang and Liu gave characterization of the compact generalized composition operators from Bloch type spaces to \mathcal{Q}_K type spaces in terms of K -Carleson measure in [14]. Essential norm

of generalized composition operators from weighted Dirichlet or Bloch type spaces to \mathcal{Q}_K type spaces was studied in [8]. A characterization of boundedness and compactness of generalized composition and Volterra type operators between \mathcal{Q}_K spaces was provided in [7]. In this paper, we determine conditions under which the generalized composition operator C_φ^g from logarithmic Bloch type spaces to \mathcal{Q}_K type spaces is bounded or compact without using Carleson measure. In this paper constants are denoted by C , they are positive and not necessarily the same in each occurrence.

2 Main results

Note that, if $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p, q)$, then $C_\varphi^g : \mathcal{B}_{\log^\beta}^\alpha \rightarrow \mathcal{Q}_K(p, q)$ is bounded, by the closed graph theorem. We now give an equivalent condition for boundedness and compactness of this operator.

Theorem 2.1 *Let $\alpha, p \in (0, \infty)$, $\beta \in [0, \infty)$, $q \in (-2, \infty)$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of \mathbb{D} . Then $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p, q)$ if and only if*

$$L := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p (1 - |z|^2)^q K(G(z, a))}{(1 - |\varphi(z)|^2)^{\alpha p} \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |\varphi(z)|^2}\right)^{\beta p}} dA(z) < \infty. \quad (2.1)$$

Proof. Suppose that $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p, q)$. By [9, Theorem 3] there exist two functions $f_1, f_2 \in \mathcal{B}_{\log^\beta}^\alpha$ such that

$$\frac{C}{(1 - |z|)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2}\right)^\beta} \leq |f_1'(z)| + |f_2'(z)|, \quad z \in \mathbb{D}.$$

Using (1), we get

$$\frac{C}{(1 - |\varphi(z)|^2)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |\varphi(z)|^2}\right)^\beta} \leq |f_1'(\varphi(z))| + |f_2'(\varphi(z))|, \quad z \in \mathbb{D}.$$

It follows that

$$\begin{aligned} & \frac{C|g(z)|^p(1-|z|^2)^qK(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2}\right)^{\beta p}} \\ & \leq 2^p(|f'_1(\varphi(z))|^p+|f'_2(\varphi(z))|^p)|g(z)|^p(1-|z|^2)^qK(G(z,a)). \end{aligned}$$

Integrating with respect to z , we have

$$\int_{\mathbb{D}} \frac{|g(z)|^p(1-|z|^2)^qK(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2}\right)^{\beta p}} dA(z) \leq C(\|C_\varphi^g(f_1)\|_{\mathcal{Q}_K(p,q)}^p + \|C_\varphi^g(f_2)\|_{\mathcal{Q}_K(p,q)}^p).$$

Since $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p,q)$, the inequality (2.1) follows.

Conversely, for $f \in \mathcal{B}_{\log^\beta}^\alpha$ we have

$$\begin{aligned} \|C_\varphi^g(f)\|_{\mathcal{Q}_K(p,q)}^p &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) \\ &\leq b_{\alpha,\beta}^p(f) \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p (1-|z|^2)^q K(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p} \left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2}\right)^{\beta p}} dA(z) \\ &\leq L \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}^p, \end{aligned}$$

which implies that $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p,q)$. \square

Moreover, the above argument shows that $C_\varphi^g : \mathcal{B}_{\log^\beta}^\alpha \rightarrow \mathcal{Q}_K(p,q)$ is, infact, bounded if and only if $L < \infty$.

For example, if $q = \beta = 0$, $p = 2$, $\alpha = 1$, $K(t) = 1$, $g(z) = 1$ and $\varphi(z) = z$, then L is infinity. Note that in this case, by [13, Theorem 1.2.1] the lacunary series $f(z) = \sum_{k=0}^{\infty} z^{2^k}$ is in $\mathcal{B}_{\log^\beta}^\alpha = \mathcal{B}$ and by [11, Theorem 7], it is not in $\mathcal{Q}_K(p,q) = \mathcal{Q}_K$, hence $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p,q)$ does not hold.

By [9, Lemma 3], we have the following estimates for the growth rate of the functions f in $\mathcal{B}_{\log^\beta}^\alpha$

$$|f(z)| \leq C \begin{cases} |f(0)| + \|f\|_{\mathcal{B}_{\log^\beta}^\alpha} & \alpha \in (0, 1) \quad \text{or} \quad \alpha = 1, \beta > 1 \\ |f(0)| + \|f\|_{\mathcal{B}_{\log^\beta}^\alpha} \ln \ln \frac{e^{\frac{\beta}{1-|z|}}}{1-|z|} & \alpha = \beta = 1 \\ |f(0)| + \|f\|_{\mathcal{B}_{\log^\beta}^\alpha} (\ln \frac{e^{\frac{\beta}{1-|z|}}}{1-|z|})^{1-\beta} & \alpha = 1, \beta \in (0, 1) \\ |f(0)| + \frac{\|f\|_{\mathcal{B}_{\log^\beta}^\alpha}}{(1-|z|)^{\alpha-1} (\ln \frac{e^{\frac{\beta}{1-|z|}}}{1-|z|})^\beta} & \alpha > 1, \beta \geq 0, \end{cases} \quad (2.2)$$

for some $C > 0$ independent of f . By (1) we can replace $1 - |z|$ by $1 - |z|^2$ in (2.2).

Using (1.1), (2.2) and similar to the proof of [7, Lemma 2.1], we have the following Lemma.

Lemma 2.1 *Let $\alpha, p \in (0, \infty)$, $\beta \in [0, \infty)$, $q \in (-2, \infty)$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of \mathbb{D} such that $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p, q)$. Then C_φ^g is Compact if and only if for any bounded sequence (f_n) in $\mathcal{B}_{\log^\beta}^\alpha$ which converges to zero uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$, we have $\|C_\varphi^g(f_n)\|_{\mathcal{Q}_K(p, q)} \rightarrow 0$ as $n \rightarrow \infty$.*

By Lemma 2.1 we prove the main result of this paper.

Theorem 2.2 *Let $0 \leq \beta < \alpha < \infty$, $p \in (0, \infty)$, $q \in (-2, \infty)$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of \mathbb{D} such that $C_\varphi^g(\mathcal{B}_{\log^\beta}^\alpha) \subseteq \mathcal{Q}_K(p, q)$. Then C_φ^g is compact if and only if*

$$\limsup_{r \rightarrow 1} \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} \frac{|g(z)|^p (1 - |z|^2)^q K(G(z, a))}{(1 - |\varphi(z)|^2)^{\alpha p} (\ln \frac{e^{\frac{\beta}{1-|\varphi(z)|^2}}}{1-|\varphi(z)|^2})^{\beta p}} dA(z) = 0. \quad (2.3)$$

Proof. Let (2.3) hold and (f_n) be a sequence in the closed unit ball of $\mathcal{B}_{\log^\beta}^\alpha$ such that $f_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$.

By hypothesis for every $\varepsilon > 0$, there is $\delta \in (0, 1)$ such that

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > \delta} \frac{|g(z)|^p (1 - |z|^2)^q K(G(z, a))}{(1 - |\varphi(z)|^2)^{\alpha p} \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |\varphi(z)|^2}\right)^{\beta p}} dA(z) < \varepsilon.$$

Let $\Delta = \{w \in \mathbb{D} : |w| \leq \delta\}$. Then

$$\begin{aligned} \|C_\varphi^g(f_n)\|_{\mathcal{Q}_K(p,q)}^p &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'_n(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) \\ &= \sup_{a \in \mathbb{D}} \left[\int_{|\varphi(z)| \leq \delta} |f'_n(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) \right. \\ &\quad \left. + \int_{\delta < |\varphi(z)| < 1} |f'_n(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) \right] \\ &\leq \sup_{w \in \Delta} |f'_n(w)|^p \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| \leq \delta} |g(z)|^p (1 - |z|^2)^q \\ &\quad \times K(G(z, a)) dA(z) \\ &\quad + b_{\alpha,\beta}^p(f_n) \sup_{a \in \mathbb{D}} \int_{\delta < |\varphi(z)| < 1} \frac{|g(z)|^p (1 - |z|^2)^q K(G(z, a))}{(1 - |\varphi(z)|^2)^{\alpha p} \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |\varphi(z)|^2}\right)^{\beta p}} dA(z) \\ &\leq \sup_{w \in \Delta} |f'_n(w)|^p \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| \leq \delta} |g(z)|^p (1 - |z|^2)^q \\ &\quad \times K(G(z, a)) dA(z) + \varepsilon, \end{aligned}$$

since $b_{\alpha,\beta}^p(f_n) \leq \|f_n\|_{\mathcal{B}_{\log \beta}^\alpha} \leq 1$. By [2, VII, Theorem 2.1], the sequence (f'_n) converges to zero uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$. In particular, $\sup_{w \in \Delta} |f'_n(w)|^p \rightarrow 0$ as $n \rightarrow \infty$. Hence the boundedness of the kernel function K and the boundedness of g on the compact subset $\{z : |\varphi(z)| \leq \delta\}$ of \mathbb{D} implies that $\|C_\varphi^g(f_n)\|_{\mathcal{Q}_K(p,q)} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, by Lemma 2.1, $C_\varphi^g : \mathcal{B}_{\log \beta}^\alpha \rightarrow \mathcal{Q}_K(p, q)$ is compact.

Conversely, let C_φ^g be compact. Since

$$\begin{aligned} b_{\alpha,\beta} \left(\frac{z^n}{n^{1-\alpha+\beta}} \right) &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2} \right)^\beta n^{\alpha-\beta} |z|^{n-1} \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \left(\frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2} \right)^\beta n^{\alpha-\beta} |z|^{n-1} \\ &\leq C \sup_{z \in \mathbb{D}} (1 - |z|)^{\alpha-\beta} n^{\alpha-\beta} |z|^{n-1}, \end{aligned}$$

and for $0 \leq \beta < \alpha < \infty$, $f(x) = n^{\alpha-\beta}x^{n-1}(1-x)^{\alpha-\beta}$ has a maximum in $\frac{n-1}{n-1+\alpha-\beta}$, the sequence $(\frac{z^n}{n^{1-\alpha+\beta}})$ is norm bounded in $\mathcal{B}_{\log^\beta}^\alpha$. It is well known that the series $\sum_{n=1}^\infty \frac{r^n}{n^{1-\alpha+\beta}}$ converges for any $r \in (0, 1)$. Hence the sequence $(\frac{z^n}{n^{1-\alpha+\beta}})$ converges to zero uniformly on compact subsets of \mathbb{D} , using Lemma 2.1, we have $\|C_\varphi^g(\frac{z^n}{n^{1-\alpha+\beta}})\|_{\mathcal{Q}_K(p,q)} \rightarrow 0$ as $n \rightarrow \infty$. Whence for given $\varepsilon > 0$,

$$n^{(\alpha-\beta)p} \int_{\mathbb{D}} |\varphi(z)|^{p(n-1)} |g(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) < \varepsilon,$$

for large enough n . Thus for each $r \in (0, 1)$,

$$n^{(\alpha-\beta)p} r^{p(n-1)} \int_{|\varphi(z)|>r} |g(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) < \varepsilon.$$

Taking $r \geq n^{\frac{\beta-\alpha}{n-1}}$, we obtain

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} |g(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) < \varepsilon. \quad (2.4)$$

On the other hand, for any f in the closed unit ball $\mathbb{B}_{\mathcal{B}_{\log^\beta}^\alpha}$ of $\mathcal{B}_{\log^\beta}^\alpha$, if we set $f_t(z) = f(tz)$, then $f_t \rightarrow f$ uniformly on compact subsets of \mathbb{D} as $t \rightarrow 1$. Since $C_\varphi^g : \mathcal{B}_{\log^\beta}^\alpha \rightarrow \mathcal{Q}_K(p,q)$ is compact, using Lemma 2.1, $\|C_\varphi^g(f_t - f)\|_{\mathcal{Q}_K(p,q)} \rightarrow 0$ as $t \rightarrow 1$. Let $\varepsilon > 0$ be given. Choose $t \in (0, 1)$ such that

$$\int_{\mathbb{D}} |(C_\varphi^g f_t)'(z) - (C_\varphi^g f)'(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) < \varepsilon.$$

Using this inequality along with (2.3), we have

$$\begin{aligned} & \int_{|\varphi(z)|>r} |(C_\varphi^g f)'(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) \\ & \leq C(\varepsilon + \int_{|\varphi(z)|>r} |(C_\varphi^g f_t)'(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z)) \\ & \leq C\varepsilon(1 + \sup_{z \in \mathbb{D}} |f_t'(z)|^p). \end{aligned}$$

Thus for every $f \in \mathbb{B}_{\mathcal{B}_{\log\beta}^\alpha}$ and every $\varepsilon > 0$, there exists a $\delta = \delta(f, \varepsilon)$ such that

$$\int_{|\varphi(z)|>r} |(C_\varphi^g f)'(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) < \varepsilon, \quad (2.5)$$

for all $r \in [\delta, 1)$. As mentioned in the previous theorem, there are two functions $f_1, f_2 \in \mathcal{B}_{\log\beta}^\alpha$ such that for each $z \in \mathbb{D}$,

$$\frac{C}{(1 - |z|^2)^\alpha (\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |z|^2})^\beta} \leq |f_1'(z)| + |f_2'(z)|.$$

Let $\delta = \max_{1 \leq k \leq 2} \delta(\frac{f_k}{\|f_k\|}, \varepsilon)$ and using (2.5) then we have

$$\begin{aligned} 2\varepsilon &> \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \frac{1}{\|f_1\|_{\mathcal{B}_{\log\beta}^\alpha}^p} |(C_\varphi^g f_1)'(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) \\ &\quad + \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \frac{1}{\|f_2\|_{\mathcal{B}_{\log\beta}^\alpha}^p} |(C_\varphi^g f_2)'(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) \\ &\geq C \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} (|f_1'(\varphi(z))|^p + |f_2'(\varphi(z))|^p) |g(z)|^p (1 - |z|^2)^q K(G(z, a)) dA(z) \\ &\geq C \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \frac{|g(z)|^p (1 - |z|^2)^q K(G(z, a))}{(1 - |\varphi(z)|^2)^{\alpha p} (\ln \frac{e^{\frac{\beta}{\alpha}}}{1 - |\varphi(z)|^2})^{\beta p}} dA(z), \end{aligned}$$

for all $r \in [\delta, 1)$, which implies (2.3). \square

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