Mathematics Scientific Journal
Vol. 8, No. 1, (2012), 15-21

Limit summability of ultra exponential functions

M.H. Hooshmand [a,*]

[a] *Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran*

Received 12 March 2011; accepted 1 January 2012

Abstract

In [1] we uniquely introduced ultra exponential functions (uxp_a) and defined next step of the serial binary operations: addition, multiplication and power. Also, we exhibited the topic of limit summability of real functions in [2,3]. In this paper, we study limit summability of the ultra exponential functions and prove some of their properties. Finally, we pose an unsolved problem about them.

Key words: Limit summable function, ultra exponential function, ultra power, limit summand function.

2010 AMS Mathematics Subject Classification : 33E30; 33B99; 40A05.

* Corresponding author's E-mail: hadi.hooshmand@gmail.com, hooshmand@iaushiraz.ac.ir(M. H. Hooshmand), Tel: +98-9366244904; Fax:+98-7116402488

1 Introduction and Preliminaries

In [1] we introduced the ultra exponential functions as follows. Let a be a positive real number and n a natural number. We call the notation $a^{\underline{n}}$, *a to the ultra power of* n and define by the following recursive definition

$$a^{\underline{n}} = a^{a^{\underline{n-1}}} \quad : \quad \text{for } n = 2, 3, \cdots \quad , \; a^{\underline{1}} = a.$$

Let apply the above relation for $n = 1$, then $a = a^{\underline{1}} = a^{a^{\underline{0}}}$, so we define $a^{\underline{0}} = 1$. Putting $n = 0$ implies that $a^{\underline{-1}} = 0$. But it is impossible to take $n = -1$ and so $a^{\underline{-2}}, a^{\underline{-3}}, \ldots$ are not well-defined.
As we know due to algebraic properties of power, it can be extend from natural to rational numbers (and then other numbers) but this can not be done for ultra power. Because it has no any useful algebraic properties except $a^{\underline{n}} = a^{a^{\underline{n-1}}}$. Perhaps it is the reason that ultra power is not extended so far. For removing this problem we first introduced the following equation (ultra exponential functional equation) in [1] and thereafter studied some of its properties

$$f(x) = a^{f(x-1)}, \;\; f(0) = 1.$$

Finally, we proved a uniqueness theorem for the functional equation and obtained the ultra exponential functions (uxp_a) by the equation: $\mathrm{uxp}_a(x) = \exp_a^{[x]}(a^{(x)})$ where $[x]$ is the largest integer not exceeding x, $(x) = x - [x]$ and $0 < a \neq 1$, $\exp_a(x) = a^x$, $\exp(x) = \exp_e(x) = e^x$ (for all x).
Also, in [2] we introduced the topic of *Limit summability of real functions* as a generalization of the Gamma-type functions discussed in [4]. The followings are its summary.
Let $f : D_f \to \mathbb{R}$, where $\mathbb{N} \subseteq D_f \subseteq \mathbb{R}$. For a function with domain D_f, we put $\Sigma_f = \{x | x + \mathbb{N} \subseteq D_f\}$, and then for any $x \in \Sigma_f$ and positive integer n set

$$\begin{aligned} R_n(f, x) &= R_n(x) = f(n) - f(x+n), \\ f_{\sigma_n}(x) &= \sigma_n(f, x) = xf(n) + \sum_{k=1}^{n} R_k(x). \end{aligned}$$

When $x \in D_f$, we may use the notation $\sigma_n(f(x))$ instead of $\sigma_n(f, x)$.

The function f is called limit summable at $x_0 \in \Sigma_f$ if the functional sequence $\{f_{\sigma_n}(x)\}$ is convergent at $x = x_0$. The function f is called limit summable on the set $S \subseteq \Sigma_f$ if it is limit summable at all the points of S.
Now, put $D_{f_\sigma} = \{x \in \Sigma_f | f$ is summable at $x\}$. The function f_σ is the same limit function f_{σ_n} with domain D_{f_σ}. We represent also, the limit function $R_n(f, x)$ as $R(f, x)$ or $R(x)$.
The function f is called limit summable if it is summable on Σ_f, $R(1) = 0$ and $D_f \subseteq D_f - 1$. In this case the function f_σ is referred to as the limit summand function of f. If f is limit summable, then $D_{f_\sigma} = D_f - 1$ and

$$f_\sigma(x) = f(x) + f_\sigma(x-1); \quad \forall x \in D_f \tag{1.1}$$

Therefore

$$f_\sigma(m) = f(1) + \cdots + f(m) = \sum_{j=1}^{m} f(j); \quad \forall m \in \mathbb{N}^*.$$

The function f is called *weak limit summable* if $\Sigma_f = D_{f_\sigma}$. Moreover, if $R(1) = 0$, then f is called *semi limit summable.* Very often if f is limit summable on an interval of length 1 and $R(1) = 0$, then f is summable (see [2,3]).

Example 1.1 *If $0 < b \neq 1$ and $0 < a < 1$, then the real function $f(x) = ca^x + \log_b x$ is limit summable and*

$$f_\sigma(x) = \frac{ca}{a-1}(a^x - 1) + \log_b \Gamma(x+1).$$

2 Main Theorems

At the firs we restate the uniqueness theorem for ultra exponential functions.

Theorem A. Let $0 < a \neq 1$. If $f : (-2, +\infty) \to \mathbb{R}$ satisfies the following conditions

(i)

$$f(x) = a^{f(x-1)} \quad \text{for all} \;\; x > -1, \;\; f(0) = 1,$$

(ii) f is differentiable on $(-1, 0)$ and f' is a nondecreasing or nonincreasing function on $(-1, 0)$,
(iii)

$$\lim_{x \to 0^+} f'(x) = (\ln a) \lim_{x \to 0^-} f'(x),$$

then f is uniquely determined through the equation

$$f(x) = \exp_a^{[x]}(a^{(x)}) = \exp_a^{[x+1]}((x)) = \begin{cases} \vdots & \vdots \\ \log_a(x+2) & -2 < x < -1 \\ 1 + x & -1 \leq x < 0 \\ a^x & 0 \leq x < 1 \\ a^{a^{x-1}} & 1 \leq x < 2 \\ \vdots & \vdots \end{cases}$$

Proof. See [1]. □

Thus, for example, we have $2^{\underline{4}} = 65536$, $e^{\underline{\pi/2}} = 5.868...$, $0.5^{\underline{-4.3}} = 4.03335...$, $0.6^{\underline{-5.264}} = -5.35997...$, $0.7^{\underline{3.1}} = 0.7580...$.
Therefore,

$$a^{\underline{x}} = a^{a^{\underline{x-1}}} \quad \text{for all } x > -1,$$

the function uxp_a is continuous on $(-2, +\infty)$ and differentiable on $(-2, +\infty) \setminus \mathbb{Z}$, and if $a > 1$ then, uxp_a is increasing. Also, if $0 < a < 1$, then $a^{\underline{2n-1}}$ is increasing, $a^{\underline{2n}}$ is decreasing and we have

$$0 \leq a^{\underline{2m-1}} < \alpha_1 \leq \alpha_2 < a^{\underline{2k}} \leq 1 \quad : \quad \forall m, k \in \mathbb{N},$$

where $\alpha_1 \leq \alpha_2$ are the zeros of $a^{a^x} - x = 0$. It was shown that $\alpha_1 = \alpha_2$ if and only if $(\frac{1}{e})^e \leq a < 1$, and in the case $a > 1$, $a^{\underline{n}}$ is bounded above if and only if $1 < a < e^{(\frac{1}{e})}$. Considering these facts, we proved that (see [1]) :

$$\lim_{x \to +\infty} a^{\underline{x}} = \begin{cases} +\infty \; ; & a > e^{\frac{1}{e}} \\ \alpha \; ; & (\frac{1}{e})^e \leq a \leq e^{(\frac{1}{e})} \\ \text{does not exist} \; ; & 0 < a < (\frac{1}{e})^e \end{cases}$$

where α is the least zero of the equation $a^x - x = 0$. Now, we want to study limit summability of ultra exponential functions (uxp_a) and determine their properties with resect to range values of a.

Theorem 2.1 *Suppose that* $0 < a \neq 1$ *and consider the ultra exponential function* uxp_a.
(i) If $0 < a < (\frac{1}{e})^e$*, then* uxp_a *is not (weak) limit summable.*
(ii) If $\frac{1}{e} \leq a \leq e^{\frac{1}{e}}$*, then* uxp_a *is limit summable.*
(iii) If $a > e^{\frac{1}{e}}$*, then* uxp_a *is not (weak) limit summable.*

Proof. First, note that if $0 < a < (\frac{1}{e})^e$, then $\lim_{n\to\infty} a^{\underline{2n-1}} = \alpha_1 < \alpha_2 = \lim_{n\to\infty} a^{\underline{2n}}$, where $\alpha_1 < \alpha_2$ are the roots of $a^{a^x} - x$. Thus

$$\lim_{n\to\infty} R_{2n-1}(\mathrm{uxp}_a, 1) = \lim_{n\to\infty}(a^{\underline{2n-1}} - a^{\underline{2n}}) = \alpha_1 - \alpha_2 < 0$$

$$\lim_{n\to\infty} R_{2n}(\mathrm{uxp}_a, 1) = \lim_{n\to\infty}(a^{\underline{2n}} - a^{\underline{2n+1}}) = \alpha_2 - \alpha_1 > 0$$

Also, if $a > e^{\frac{1}{e}}$, then $\lim_{n\to\infty} a^{\underline{n}} = \infty$ so $\lim_{n\to\infty} R_n(\mathrm{uxp}_a, 1) = \lim_{n\to\infty}(a^{\underline{n}} - a^{a^{\underline{n}}}) = -\infty$. Therefore, in the cases (i) and (iii) $R_n(\mathrm{uxp}_a, 1)$ is not convergent and so uxp_a is not weak limit summable.
Now, fix $0 < x < 1$ and consider the following two cases:
Case 1) $1 < a \leq e^{\frac{1}{e}}$, then $a^{\underline{k}} < a^{\underline{k+x}} < a^{\underline{k+1}}$, for every positive integer k, so

$$a^{\underline{k}} - a^{\underline{k+1}} < a^{\underline{k}} - a^{\underline{k+x}} < 0 \quad : \quad \forall k \in \mathbb{N}.$$

Since

$$\sum_{k=1}^{\infty}(a^{\underline{k}} - a^{\underline{k+1}}) = a - \lim_{n\to\infty} a^{\underline{n}} = a - \alpha,$$

then $\sum_{k=1}^{\infty}(a^{\underline{k}} - a^{\underline{k+x}})$ is convergent so uxp_a is limit summable at x and

$$a + \alpha(x-1) \leq (\mathrm{uxp}_a)_\sigma(x) \leq \alpha x \quad : \quad \forall x \in [0,1] \tag{2.1}$$

Case 2) $\frac{1}{e} \leq a < 1$. We have

$$(-1)^{k+1} a^{\underline{k}} < (-1)^{k+1} a^{\underline{k+x}} < (-1)^{k+1} a^{\underline{k+1}} \quad : \quad \forall k \in \mathbb{N} \tag{2.2}$$

so $\sum_{k=1}^{\infty}(a^{\underline{k}} - a^{\underline{k+x}})$ is an alternating series. Since $a^x + x$ is an increasing function on $[\frac{\ln\ln a^{-1}}{\ln a^{-1}}, +\infty) \supseteq [0,\infty)$ (because $\frac{1}{e} \leq a < 1$) , then (2.2) implies that

$$|a^{\underline{k}} - a^{\underline{k+x}}| > |a^{\underline{k+1}} - a^{\underline{k+1+x}}| \quad : \quad \forall k \in \mathbb{N}.$$

Also, $\lim_{k\to\infty} a^{\underline{k}} - a^{\underline{k+x}} = \alpha - \alpha = 0$. Therefore, the series is convergent, by the Leibniz criterion, and so uxp_a is limit summable at x.
Finally, Theorem 2.11 of [2] grantees that uxp_a is limit summable in both of the two cases and so the proof is complete. □

Remark 2.1 *For the case $(\frac{1}{e})^e \le a < \frac{1}{e}$, if $a^{\underline{k}} \ge \frac{\ln\ln a^{-1}}{\ln a^{-1}}$ for all positive integers k from a number on (e.g. $ln\frac{1}{a} \le (\frac{1}{a})^a$ that implies $a^{\underline{k}} \ge \frac{\ln\ln a^{-1}}{\ln a^{-1}}$ for every $k \ge 1$), then we can say uxp_a is limit summable (considering the above proof). So, we leave an unsolved problem at the end of paper.*

Denote the summand function of $a^{\underline{x}}$ by $\overline{\sigma}_a(x)$, hence

$$\overline{\sigma}_a(x) = (\mathrm{uxp}_a)_\sigma(x) = \alpha x + \sum_{k=1}^{\infty}(a^{\underline{k}} - a^{\underline{k+x}}) \quad : \quad \forall x > -3.$$

Some properties of $\overline{\sigma}_a$ are as following:

$$\overline{\sigma}_a(x) = a^{\underline{x}} + \overline{\sigma}_a(x-1) \quad : \quad \forall x > -2,$$

$$\overline{\sigma}_a(n) = a^{\underline{1}} + \cdots + a^{\underline{n}} = \sum_{k=1}^{n} a^{\underline{k}} \quad : \quad \forall n \in \mathbb{N}.$$

Theorem 2.2 *If $1 < a \le e^{\frac{1}{e}}$, then $\lim_{x\to+\infty} \overline{\sigma}_a(x) = +\infty$.*

Proof. The above equality implies that

$$\overline{\sigma}_a(x) = \overline{\sigma}_a((x)) + \sum_{k=1}^{[x]} a^{\underline{(x)+k}} \quad : \quad \forall x > 0.$$

Since $a^{\underline{t}}$ is increasing, then $a^{\underline{k}} < a^{\underline{x+k}}$ for all $x > 0$ and natural k. On the other hand $\overline{\sigma}_a((x)) \ge a - \alpha$, by (2.1). Therefore

$$\overline{\sigma}_a(x) \ge a - \alpha + \sum_{k=1}^{[x]} a^{\underline{k}} \quad : \quad \forall x > 0.$$

So $\lim_{x\to+\infty}\overline{\sigma}_a(x)=+\infty$, because $\sum_{k=1}^{\infty}a^{\underline{k}}=+\infty$ (note that $a^{\underline{k}}\to\alpha>0$). $\square$

Question. Is uxp_a limit summable for $(\frac{1}{e})^e\le a<\frac{1}{e}$? If no, what is the largest subset of $\mathbb{R}$ such that uxp_a is limit summable on it?
(note that it is at least limit summable on $\mathbb{N}$, because $R(\mathrm{uxp}_a,1)=0$).

Acknowledgement. This work is supported by the Research Foundation of Islamic Azad University - Shiraz Branch.
Also, the author would like to thank the referees for helpful suggestions and comments.

References

[1] M.H. Hooshmand, *Ultra Power and Ultra Exponential Functions*, Integral Transforms Spec. Funct., Vol. 17, No. 8 (2006), 549-558.

[2] M.H. Hooshmand, *Limit Summability of Real Functions*, Real Analysis Exchange, Volume 27 (2002), Number 2, Pages 463-472.

[3] M.H. Hooshmand, *Another Look at the Limit Summability of Real Functions*, J. of Math. Extension, Vol. 4, No. 1 , Ser. No. 7 (2009), 73-89.

[4] R. J. Webster, *Log-Convex Solutions to the Functional Equation $f(x+1)=g(x)f(x)$: Γ -Type Functions*, J. Math. Anal. Appl., 209 (1997), 605-623.