Mathematics Scientific Journal
Vol. 9, No. 1, (2013), 105-113

A note on positive definiteness and stability of interval matrices

H. Veiseh [a,*]

[a] *Department of Applied Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran*

Received 17 June 2013; accepted 28 February 2013

Abstract

It is proved that by using bounds of eigenvalues of an interval matrix, some conditions for checking positive definiteness and stability of interval matrices can be presented. These conditions have been proved previously with various methods and now we provide some new proofs for them with a unity method. Furthermore we introduce a new necessary and sufficient condition for checking stability of interval matrices.

Key words: Interval matrix; Real eigenvalues; Positive definiteness; Stability; Symmetric matrix.

2010 AMS Mathematics Subject Classification : 15B48; 65G40.

---

* Corresponding author's E-mail: veisehana@yahoo.com(H.Veiseh)

1 Introduction

It is well-known that positive definiteness and stability of a given point matrix can be inferred if one knows signs of its eigenvalues. In other word, if all eigenvalues are positive, then the given matrix is positive definite; and if their real parts are negative, it is stable- in the sense of finding solution for differential equations, so knowing the signs of all eigenvalues has a practical importance. On the other hand, finding all eigenvalues of a given matrix is not an easy job in general. consequently, finding their signs without computing them, clear the problem of positive definiteness as well as stability. This problem have been well studied in scalar case and for its interval case, there are some valuable research too which we mention in the following.

First of all, checking positive definiteness and stability of interval matrices are known to be NP-hard problems [1,2]. Up to know several conditions have been introduced to verify positive definiteness and stability of interval matrices [2,3,4,5,6]. The problem of computing eigenvalues of interval matrices is known to be NP-hard too, even checking whether zero is an eigenvalue of an interval matrix is an NP-hard problem, since it is equivalent to checking regularity of the interval matrix, which is NP-hard [2,7]. The first results about eigenvalues was probably due to Deif [8]. Indeed the problem of computing lower and upper bounds for the eigenvalue set, which we focused on, is well studied [9,10,11,12].

The main object of the current work is to use the discovered bounds on eigenvalues of interval matrices to proof some conditions for checking positive definiteness and stability of interval matrices.

The structure of this paper is as follows: After presenting some facts and theorems about real eigenvalues and extremal bounds of them, we continue our paper in two folds, positive definiteness and stability of symmetric interval matrices. In positive definiteness part, section 3, we use lower bound of real eigenvalues to prove conditions for checking positive definiteness of interval matrices. In stability part, section 4, we use the upper bound of real eigenvalues and some other facts to prove two conditions for checking stability, which one of them, Theorem 4.2, is new and obtained from using our method.

2 Basic results and notations

Let us introduce some notation: An interval matrix is defined as

$$\mathbf{A} = \{A \in \mathbb{R}^{m\times n} | \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}, \underline{A} \leq \overline{A}$, are given matrices. By

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}), \qquad \Delta = \frac{1}{2}(\underline{A} - \overline{A}),$$

we denote the midpoint and the radius of $\mathbf{A}$, respectively.
Notice that in this paper we focus on symmetric interval matrices, so first we give its definition:

Definition 2.1 *A square interval matrix is called symmetric if* $\mathbf{A}^T = \mathbf{A}$*, where*

$$\mathbf{A}^T = \{A^T : A \in \mathbf{A}\}.$$

It is clear that $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *is symmetric if and only if both* A_c *and* Δ *are symmetric, but generally a symmetric interval matrix may contain nonsymmetric point matrices.*

First we restate the main and basic property of symmetric scalar (point) matrices, which plays an essential role in all over discussions in this paper.

Remark 2.1 *A symmetric matrix* $A \in \mathbb{R}^{n\times n}$ *has all eigenvalues real. They are usually ordered in a nonincreasing sequence as*

$$\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A),$$

where $\lambda_i(A)$, $i = 1, \ldots, n$, are eigenvalues of A and we denote minimal and maximal eigenvalue of a symmetric matrix by $\lambda_{\min}$ (or λ_n) and $\lambda_{\max}$ (or λ_1) respectively.

Definition of eigenvalues of an interval matrix is given as follows.

Definition 2.2 *Let $\mathbf{A}$ be a square interval matrix. Its real eigenvalue set is defined as*

$$\Lambda(\mathbf{A}) := \{\lambda \in \mathbb{R}; Ax = \lambda x, x \neq 0, A \in \mathbf{A}\}.$$

In other words, a real number λ is called a real eigenvalue of $\mathbf{A}$ if it is a real eigenvalue of some $A \in \mathbf{A}$.

If $\mathbf{A}$ is symmetric, then for each $i \in \{1, \ldots, n\}$ the set

$$[\underline{\lambda}_i(\mathbf{A}), \overline{\lambda}_i(\mathbf{A})] = \{\lambda_i(A) | A \in \mathbf{A}, A\, symmetric\},$$

is a compact interval. The set of all possible eigenvalues forms a union of the mentioned compact real interval. Our focus in this paper is on extremal, minimum and maximum, eigenvalues. The following theorem introduces the upper and lower bounds for eigenvalues of an interval matrix and is a very useful means in proving theorems about positive definiteness and stability [10].

Theorem 2.1 *For each symmetric $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$, there holds*

$$\overline{\lambda}_1(\mathbf{A}) = \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|), \tag{2.1}$$

$$\underline{\lambda}_n(\mathbf{A}) = \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|). \tag{2.2}$$

On the other hand we could reformulate the problem and compute enclosure of the intervals $[\underline{\lambda}_i(\mathbf{A}), \overline{\lambda}_i(\mathbf{A})]$, $i = 1, \ldots, n$, as follows [9,10].

Theorem 2.2 *For a symmetric $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ we have*

$$[\underline{\lambda}_i(\mathbf{A}), \overline{\lambda}_i(\mathbf{A})] \subseteq [\lambda_i(A_c) - \varrho(\Delta), \lambda_i(A_c) + \varrho(\Delta)].$$

Corollary 2.1 *In particular for each $\lambda_i(A)$, eigenvalue of each symmetric $A \in \mathbf{A}$ there holds*

$$\lambda_n(A_c) - \varrho(\Delta) \leq \lambda_i(A) \leq \lambda_1(A_c) + \varrho(\Delta).$$

3 Positive definiteness of interval matrices

Definition 3.1 *A symmetric interval matrix is called to be positive definite if each symmetric $A \in \mathbf{A}$ is positive definite.*

In our discussion, the next well-known Lemma has a significant role [13].

Lemma 3.1 *A symmetric point matrix is positive definite if and only if all its eigenvalues are positive.*

By Corollary 2.1, we can prove the following practical criterion for checking positive definiteness which was proved by Rohn [6].

Theorem 3.1 *A symmetric interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is positive definite if*

$$\lambda_{\min} A_c = \lambda_n(A_c) > \varrho(\Delta).$$

Proof. Let $\lambda_n(A_c) - \varrho(\Delta) > 0$, using Corollary 2.1, then $\lambda_i(A) > 0$ for each symmetric $A \in \mathbf{A}$. Therefore each symmetric $A \in \mathbf{A}$ is positive definite due to Lemma 3.1 and this implies positive definiteness of $\mathbf{A}$. □

The following theorem was introduced by Rohn [5]; then it was restated and proved again by Farhadsefat et al [3] using reductio ad absurdum. Here it is reproved directly with a novel method.

Theorem 3.2 *A symmetric interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$*

is positive definite if and only if

$$x^T A_c x - |x|^T \Delta |x| > 0$$

holds for each $x \neq 0$.

Proof. If $\mathbf{A}$ is positive definite, according to Definition 3.1, each symmetric point matrices belongs to $\mathbf{A}$ is positive definite, thus all their eigenvalues are positive. Because of (2.2) we have,

$$\underline{\lambda}_n(\mathbf{A}) = \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|) > 0.$$

Therefore

$$(x^T A_c x - |x|^T \Delta |x|) > 0.$$

Conversely if $(x^T A_c x - |x|^T \Delta |x|) > 0$, then $\min_{\|x\|_2=1}(x^T A_c x - |x|^T \Delta |x|) > 0$, hence $\underline{\lambda}_n(\mathbf{A}) > 0$. In other words, each eigenvalue of symmetric point matrix in $\mathbf{A}$ is positive. This completes the proof. □

4 Stability of interval matrices

Definition 4.1 *A square point matrix A is called Hurwitz stable if $Re\lambda < 0$ for each eigenvalue of A.*

If A is symmetric, then λ is real, also we denote real part of λ by $Re(\lambda)$.

Definition 4.2 *A square interval matrix $\mathbf{A}$ is called stable(sometimes, Hurwitz stable) if each $A \in \mathbf{A}$ is stable.*

A practical condition for checking stability is presented below whose proof is analogous with what Rohn has mentioned in [2], but now, in order to observe harmony, we rewrite it with our unity method:

Theorem 4.1 *A symmetric interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is stable if*

$$\lambda_{\max}(A_c) + \varrho(\Delta) < 0.$$

Proof. Suppose $\lambda_{\max}(A_c) + \varrho(\Delta) < 0$, from Bendixson theorem [14], for each λ of each $A \in \mathbf{A}$, we have

$$Re\lambda \leq \lambda_{\max}(\frac{1}{2}(A + A^T)), \tag{4.1}$$

since $\frac{1}{2}(A + A^T) \in \mathbf{A}$ is symmetric, so by Corollary 2.1

$$\lambda_{\max}(\frac{1}{2}(A + A^T)) \leq \lambda_{\max}(A_c) + \varrho(\Delta). \tag{4.2}$$

Combining relations (4.1) and (4.2), result

$$Re\lambda \leq \lambda_{\max}(A_c) + \varrho(\Delta) < 0 \tag{4.3}$$

hence, $\mathbf{A}$ is stable. □

The last condition is new and obtained by using our method.

Theorem 4.2 *A symmetric interval matrix is stable if and only if*

$$x^T A_c x + |x|^T \Delta |x| < 0$$

holds for each $x \neq 0$.

Proof. Let $x^T A_c x + |x|^T \Delta |x| < 0$, similar to the proof of Theorem 4.1 and due to (2.1) we have

$$Re\lambda \leq \max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta |x|) < 0,$$

and this implies stability of $\mathbf{A}$.
Conversely let $\mathbf{A}$ be stable then $Re\lambda < 0$ for each $A \in \mathbf{A}$, therefore the maximum of eigenvalues is negative too and due to (2.1),

$$\max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta |x|) < 0,$$

so $(x^T A_c x + |x|^T \Delta |x|) < 0$. □

Acknowledgment

The author wishes to express her thanks to Professor Jiri Rohn and Dr. Taher Lotfi for several helpful comments during the preparation of the paper.

References

[1] J. Rohn, Checking positive definiteness or stability of symmetric interval matrices is NP-hard, *Commentationes Mathematicae Universitatis Carolinae.* **35** (1994) 795–797.

[2] J. Rohn, Checking properties of interval matrices, *Technical Report 686, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague,* September 1996.

[3] R. Farhadsefat, T. Lotfi, J. Rohn, A note on regularity and positive definiteness of interval matrices, *Cent. Eur. J. Math.* **10**(1) (2012) 322–328.

[4] M. Mansour, Robust stability of interval matrices, *Proceeding of the 28th Conference on Decision and Control, Tampa, FL.* (1989) 46–51.

[5] J. Rohn, A Handbook of Results on Interval Linear Problems, *Prague: Czech Academy of Sciences,* 2005.

[6] J. Rohn, Positive definiteness and stability of interval matrices, *SIAM J. Matrix Anal. Appl.* **15**(1) (1994) 175–184.

[7] S. Poljak, J. Rohn, Checking roboust nonsingularity is NP-hard, *Math. Control Signals Syst.* **6**(1) (1993) 1–9.

[8] A. S. Deif, The interval eigenvalue problem, *Z. Angew. Math. Mech.* **71**(1) (1991) 61–64.

[9] M. Hladik, D. Daney, E. P. Tsigaridas, Bounds on real eigenvalues and singular values of interval matrices, *SIAM J. Matrix Anal. Appl.* **31**(4) (2010) 2116–2129.

[10] J. Rohn, Bounds on eigenvalues of interval matrices, *ZAMM, Z. Angew. Math. Mech.* **78**(Suppl. 3) (1998) S1049–S1050.

[11] J. Rohn, Interval matrices: Singularity and real eigenvalues, *SIAM Journal on Matrix Analysis and Applications.* **14**(1) (1993) 82–91.

[12] J. Rohn, A. Deif, On the range of eigenvalues of an interval matrix, *Comput.* **47**(3-4) (1992) 373–377.

[13] R. A. Horn and C. R. Johnson, Matrix Analysis, *Cambridge: Cambridge University Press,* 1985.

[14] J. Stoer and R. Bulrisch, Introduction to Numerical Analysis, *Springer-Verlag, Berlin,* 1980.