



Homological dimensions of complexes of R -modules

N. Tayarzadeh^a E. Hosseini^{a,*} Sh. Niknejad^a

^a*Department of Mathematics, Islamic Azad University, Gachsaran branch,
Gachsaran, Iran.*

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Abstract

Let R be an associative ring with identity, $\mathbf{C}(R)$ be the category of complexes of R -modules and $\text{Flat}(\mathbf{C}(R))$ be the class of all flat complexes of R -modules. We show that the flat cotorsion theory $(\text{Flat}(\mathbf{C}(R)), \text{Flat}(\mathbf{C}(R))^\perp)$ have enough injectives in $\mathbf{C}(R)$. As an application, we prove that for each flat complex \mathbf{F} and each complex \mathbf{Y} of R -modules, $\text{Ext}^i(\mathbf{F}, \mathbf{Y}) = 0$, whenever R is n -perfect and $i > n$.

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* Corresponding author's Email: esmaeilmath@gmail.com(E.Hosseini)

1 Introduction

In 1966 Spencer E. Dickson [2] introduced *torsion* theories for abelian categories by exploiting the Hom-functor. By replacing formally the Hom-functor with the Ext-functor one get the basic tools of a *cotorsion theory* in an abelian category, which naturally extends the *classical* cotorsion theory. The classical cotorsion theory, where it is developed in the 60s by Harrison and many other algebraists, is the pair (Torsion-free, Cotorsion). Based on this idea, in 1978 Luigi Salce [6], introduced the notion of cotorsion theories in the category of abelian groups.

The main task of Salce is a detailed description of the cotorsion theory $({}^\perp(S^\perp), S^\perp)$ cogenerated by S , where $S \subseteq \mathbb{Q}$ is a rank-1 group such that $1 \in S$, see [6]. These cotorsion theories (for any $S \subseteq \mathbb{Q}$) are called rational cotorsion theories. Salce in [6, Problem 2, p. 31] raised the question of whether rational cotorsion theories have enough projectives (injectives).

In 1998, Gobel and Shelah answered this question, see [5, Theorem 6.1]. They proved that any cotorsion theory of abelian groups, which is cogenerated by a set H of rank-1 groups, has enough injectives and projectives. Therefore rational cotorsion theories have enough injectives and projectives.

In 1981, Enochs raised the question of whether every module has flat cover. Also he proved that if a module has a flat precover, then it has a flat cover. Let \mathcal{F} stands for the class of all flat R -modules. It is easy to see that the problem of the existence of \mathcal{F} -precovers is equivalence to the problem of completeness of the cotorsion theory $(\mathcal{F}, \mathcal{F}^\perp)$.

The existence of covers and envelopes are essential tools of *relative homological algebra*. Let \mathcal{X} be a class of R -modules, which is closed under isomorphism, the completeness of the cotorsion theories $({}^\perp\mathcal{X}, ({}^\perp\mathcal{X})^\perp)$ and $({}^\perp(\mathcal{X}^\perp), \mathcal{X}^\perp)$, induces a relative homology with respect to \mathcal{X} . Hence complete cotorsion theories, provide us

to have homological algebra in Grothendieck categories.

In 2000, Eklof and Trlifaj proved that any cotorsion theory of R -modules which is cogenerated by a set, is a complete cotorsion theory, see [4]. Therefore by [7] the flat cotorsion theory $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set and hence it is complete. Thus the category of R -modules admits \mathcal{F} -covers and \mathcal{F}^\perp -envelopes.

2 Complete cotorsion theories in the category of complexes of R -modules

Throughout this section, let \mathcal{G} be a Grothendieck category with projective generator and $\mathcal{A} = \mathbf{C}(R)$ be the category of all complexes of R -modules. Let \mathcal{X} be a class of objects of \mathcal{G} such that it is closed under isomorphisms, finite direct sums and direct summands. In this section, we will give a general definition of relative homological algebra.

Definition 2.1 *The right(left) orthogonal of \mathcal{X} in \mathcal{G} is defined as*

$$\mathcal{X}^\perp = \{\mathbf{Y} \mid \text{Ext}_{\mathcal{A}}^1(\mathbf{X}, \mathbf{Y}) = 0, \forall \mathbf{X} \in \mathcal{X}\} \quad ({}^\perp\mathcal{X} = \{\mathbf{Y} \mid \text{Ext}_{\mathcal{A}}^1(\mathbf{Y}, \mathbf{X}) = 0, \forall \mathbf{X} \in \mathcal{X}\}).$$

The pair $(\mathcal{X}, \mathcal{Y})$ is said to be a cotorsion theory in \mathcal{G} if $\mathcal{X}^\perp = \mathcal{Y}$ and $\mathcal{X} = {}^\perp\mathcal{Y}$. If there exists a class \mathcal{S} of objects in \mathcal{X} such that $\mathcal{S}^\perp = \mathcal{Y}$, we say that $(\mathcal{X}, \mathcal{Y})$ is cogenerated by \mathcal{S} .

Definition 2.2 *A cotorsion theory $(\mathcal{X}, \mathcal{Y})$ in \mathcal{G} is said to have enough injectives(projectives), if for any object \mathbf{M} of \mathcal{A} , there exists a short exact sequence*

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathbf{X} \longrightarrow \mathbf{Y} \longrightarrow 0 \quad (0 \longrightarrow \mathbf{Y} \longrightarrow \mathbf{X} \longrightarrow \mathbf{M} \longrightarrow 0)$$

for some $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$. It also called complete if it has enough injectives and projectives.

Example 2.1 *The ordinary homological algebra, induced by the complete cotorsion theories $(\text{Proj } R, R\text{-Mod})$ and $(R\text{-Mod}, \text{Inj } R)$, where $\text{Proj } R$ ($\text{Inj } R$) is the class of all projective (injective) R -modules. Those cotorsion theories have enough injectives and projectives.*

Example 2.2 *The cotorsion theory $(R\text{-Mod}, \text{Inj } R)$ is cogenerated by the set of modules R/I where I is a left ideal. Therefore it has enough injectives.*

Example 2.3 *In the classical cotorsion theory $(\text{Torsion-free}, \text{Cotorsion})$, every torsion-free abelian group J can be embedded in an exact sequence already guarantees that G is cotorsion and hence, the definition of cotorsion groups may also be given as groups G satisfying $\text{Ext}(\mathbb{Q}, G) = 0$. Therefore the classical cotorsion theory is cogenerated by the rationals \mathbb{Q} , i.e. $(\text{Torsion-free}, \text{Cotorsion}) = (\perp(\mathbb{Q}^\perp), \mathbb{Q}^\perp)$. Therefore it has enough injectives.*

Proposition 2.1 *If a cotorsion theory $(\mathcal{X}, \mathcal{Y})$ having enough injectives in \mathcal{G} , then it also have enough projectives.*

Proof. Let $(\mathcal{X}, \mathcal{Y})$ has enough injectives and \mathbf{M} be an object of \mathcal{G} . The category \mathcal{G} is a Grothendieck category with projective generators. Then there exists an exact sequence $0 \rightarrow \mathbf{T} \rightarrow \mathbf{P} \rightarrow \mathbf{M} \rightarrow 0$ with \mathbf{P} projective. By assumption, there exists an exact sequence $0 \rightarrow \mathbf{T} \rightarrow \mathbf{Y} \rightarrow \mathbf{X} \rightarrow 0$ with $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$. Using the pushout diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \mathbf{T} & \rightarrow & \mathbf{P} & \rightarrow & \mathbf{M} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & \mathbf{Y} & \rightarrow & \mathbf{Z} & \rightarrow & \mathbf{M} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbf{X} & = & \mathbf{X} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

with exact rows and columns. Since \mathcal{X} is closed under extensions, then $\mathbf{Z} \in \mathcal{X}$. Hence the middle row is the desired exact sequence.

Proposition 2.2 *Let \mathbf{S} be a nonempty subset of objects of \mathcal{A} . Then the cotorsion theory $({}^\perp(\mathbf{S}^\perp), \mathbf{S}^\perp)$ has enough injectives.*

Proof. Let \mathbf{B} be the direct sum of the objects in \mathbf{S} , \mathbf{X} be an object of \mathcal{A} and κ be an infinite cardinal number such that $\kappa > |\mathbf{B}| + |\mathbf{X}| + |R|$. Let $\beta = 2^\kappa$. By [4, Theorem 2] there exists an exact sequence $0 \rightarrow \mathbf{X} \rightarrow \mathbf{Y} \rightarrow \mathbf{K} \rightarrow 0$ of objects in \mathcal{A} such that $\text{Ext}^1(\mathbf{B}, \mathbf{Y}) = 0$. To prove that $\mathbf{K} \in {}^\perp(\mathbf{S}^\perp)$, it suffices to show that $\text{Ext}^1(\mathbf{K}, \mathbf{T}) = 0$ whenever $\text{Ext}^1(\mathbf{B}, \mathbf{T}) = 0$. However, $\mathbf{K} = \cup_{\alpha < \beta} \mathbf{K}_\alpha$ where $\mathbf{K}_\alpha = \mathbf{K}_\alpha/\mathbf{X}$, so $\mathbf{K}_0 = 0$ and for each $\alpha < \beta$, $\mathbf{K}_\alpha + 1/\mathbf{K}_\alpha \cong \mathbf{Y}_{\alpha+1}/\mathbf{Y}_\alpha \cong \mathbf{B}$. Hence, by [4, Lemma 1], $\text{Ext}^1(\mathbf{K}, X) = 0$ when $\text{Ext}^1(\mathbf{B}, X) = 0$. Then $({}^\perp(\mathbf{S}^\perp), \mathbf{S}^\perp) = ({}^\perp(\mathbf{B}^\perp), \mathbf{B}^\perp)$ has enough injectives.

3 The projective dimension of complexes of R -modules

Recall that, an acyclic complex $(\mathbf{F}, (\delta^n)_{n \in \mathbb{Z}})$ of flat R -modules is called *flat* if, for any $n \in \mathbb{Z}$, $\ker \delta^n$ is also flat R -module. We denote by $\text{Flat}(\mathbf{C}(R))$ the class of all flat complexes in $\mathcal{A} = \mathbf{C}(R)$. We will show that the cotorsion theory $(\text{Flat}(\mathbf{C}(R)), \text{Flat}(\mathbf{C}(R))^\perp)$ is a complete in \mathcal{A} . Let κ be a cardinal number such that $\kappa \geq \max\{|R|, \aleph_0\}$. Let $\mathbf{X} = (X^i, \delta_{\mathbf{X}}^i)$ and $\mathbf{Y} = (Y^i, \delta_{\mathbf{Y}}^i)$, the complex $\mathbf{Hom}^\bullet(\mathbf{X}, \mathbf{Y})$ is defined as follows:

$$\mathbf{Hom}^\bullet(\mathbf{X}, \mathbf{Y})^n = \prod_{i \in \mathbb{Z}} \text{Hom}_R(X^i, Y^{i+n})$$

and its chain map is given by

$$\delta_{\mathbf{Hom}^\bullet(\mathbf{X}, \mathbf{Y})} = \delta_{\mathbf{Y}} \circ f - (-1)^n f \circ \delta_{\mathbf{X}} (f \in \mathbf{Hom}^\bullet(\mathbf{X}, \mathbf{Y})^n).$$

Theorem 3.1 *The cotorsion theory $(\text{Flat}(\mathbf{C}(R)), \text{Flat}(\mathbf{C}(R))^\perp)$ has enough injectives.*

Proof.

Let $\mathbf{F} = (F^i, \delta^i)$ be a flat complex and $n \in \mathbb{Z}$. Let T be a subset of F^n with $|T| < \kappa$. We find a flat subcomplex $\mathbf{F}_0 = (F_0^i, \delta_0^i)$ of \mathbf{F} such that, $T \subseteq F_0^n$, $|\mathbf{F}_0| \leq \kappa$ and $\frac{\mathbf{F}}{\mathbf{F}_0}$ is a flat complex.

Without loss of generality, let $n = 0$. There exists the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_0^0 & \longrightarrow & F_0^0 & \longrightarrow & K_0^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker}(\delta^0) & \longrightarrow & F^0 & \longrightarrow & \text{Ker}(\delta^1) \longrightarrow 0, \end{array}$$

of flat modules which is pure exact in rows and columns such that, $T \subseteq F_0^0$ and $\max\{|F_0^0|, |K_0^0|, |K_0^1|\} \leq \kappa$.

We use an inductive procedure to obtain, for every $i \leq 0$, a pure exact sequences $0 \rightarrow K_0^{i-1} \rightarrow F_0^{i-1} \rightarrow K_0^i \rightarrow 0$, such that $\max\{|F_0^{i-1}|, |K_0^{i-1}|, |K_0^i|\} \leq \kappa$. Set $F_0^1 := K_0^1$, $F_0^i := 0$, for all $i > 1$, and $\delta_0^i := \delta^i|_{F_0^i}$, for all i . The complex $\mathbf{F}_0 = (F_0^i, \delta_0^i)$ provides the required complex.

Hence, for a flat complex \mathbf{F} , we can construct a continuous chain $\{\mathbf{F}_\alpha \mid \alpha \leq \gamma\}$ of flat subcomplexes of \mathbf{F} with $\mathbf{F} = \cup_{\alpha \leq \gamma} \mathbf{F}_\alpha$ such that $|\mathbf{F}_0| \leq \kappa$, for all $\alpha \leq \gamma$, $|\frac{\mathbf{F}_{\alpha+1}}{\mathbf{F}_\alpha}| \leq \kappa$ and $\frac{\mathbf{F}_{\alpha+1}}{\mathbf{F}_\alpha}$ is a flat complex. Let Y be representative set of flat complexes \mathbf{F} with $|\mathbf{F}| \leq \kappa$. Then $(\text{Flat}(\mathbf{C}(R)), \text{Flat}(\mathbf{C}(R))^\perp)$ cogenerated by Y and so by Proposition 2.2, it has enough injectives.

Corollary 3.1 *The cotorsion theory $(\text{Flat}(\mathbf{C}(R)), \text{Flat}(\mathbf{C}(R))^\perp)$ is complete.*

Proof. The result follows from Theorem 3.1 and Proposition 2.1.

Recall that a ring R is called n -perfect if $n = \sup\{\text{cd}F \mid F \text{ is a flat } R\text{-module}\} = \sup\{\text{pd}F \mid F \text{ is a flat } R\text{-module}\}$. In the remainder of this section we

let R be an n -perfect ring.

Proposition 3.1 *Let \mathbf{C} be a complex of R -modules. Then $\mathbf{C} \in \text{Flat}(\mathbf{C}(R))^\perp$ if and only if it is a complex of cotorsion R -modules.*

Proof. Let $\mathbf{C} \in \text{Flat}(\mathbf{C}(R))^\perp$. By [3], it is a complex of cotorsion R -modules and $\mathbf{Hom}^\bullet(\mathbf{F}, \mathbf{C})$ is an acyclic complex of R -modules for each flat complex \mathbf{F} .

Conversely, let \mathbf{C} be a complex of cotorsion R -modules. Then the cotorsion envelope $0 \rightarrow \mathbf{C} \rightarrow \mathbf{C}' \rightarrow \mathbf{F} \rightarrow 0$ is degree-wise split. So, \mathbf{F} be a pure acyclic complex of cotorsion flat R -modules and hence it is contractible by n -perfectness of R . Therefore $\mathbf{Hom}^\bullet(\mathbf{F}, \mathbf{C})$ is an acyclic complex for each $\mathbf{F} \in \text{Flat}(\mathbf{C}(R))$. Then by [3], $\mathbf{C} \in \text{Flat}(\mathbf{C}(R))^\perp$.

Theorem 3.2 *Let \mathbf{F} be a flat complex and \mathbf{Y} be a complex of R -modules. Then $\text{Ext}^i(\mathbf{F}, \mathbf{Y}) = 0$, for each $i > n$.*

Proof. Let \mathbf{Y} be a complex of R -modules and

$$0 \rightarrow \mathbf{Y} \rightarrow \mathbf{C}^0 \rightarrow \mathbf{C}^1 \rightarrow \dots \rightarrow \mathbf{C}^{n-1} \rightarrow \mathbf{C}^n \rightarrow \dots$$

be it's minimal cotorsion resolution by Proposition 3.1. Since $\text{Ext}^i(\mathbf{F}, \mathbf{C}^j) = 0$ for every flat complex \mathbf{F} and $i > 0$, $j \geq 0$, Then $\text{Ext}^n(\mathbf{F}, \mathbf{Y}) \cong \text{Ext}^1(\mathbf{F}, \text{Im}\delta^{n-1})$. Since R is n -perfect then $\text{Im}\delta^{n-1}$ is a complex of cotorsion R -modules and hence it belongs to $\text{Flat}(\mathbf{C}(R))^\perp$ by Proposition 3.1. Then $\text{Ext}^i(\mathbf{F}, \mathbf{Y}) = 0$, for each $i > n$.

Proposition 3.2 *The ring R is n -perfect if and only if every complex of R -modules has finite cotorsion dimension.*

Proof.

Let R be n -perfect, \mathbf{Y} be an R -module and $0 \rightarrow \mathbf{C} \rightarrow \mathbf{F}' \rightarrow \mathbf{Y} \rightarrow 0$ be the flat cover of \mathbf{Y} . Then for any flat complex \mathbf{F} we have the following exact sequence

$$0 = \text{Ext}^{n+1}(\mathbf{F}, \mathbf{C}) \rightarrow \text{Ext}^{n+1}(\mathbf{F}, \mathbf{F}') \rightarrow \text{Ext}^{n+1}(\mathbf{F}, \mathbf{Y}) \rightarrow \text{Ext}^{n+2}(\mathbf{F}, \mathbf{C}) = 0.$$

Then $\text{Ext}^{n+1}(\mathbf{F}, \mathbf{Y}) = 0$ and hence $\text{cd } \mathbf{Y} \leq n$.

The converse is trivial.

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