



## Evaluating the solution for second kind nonlinear Volterra Fredholm integral equations using hybrid method

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### Abstract

In this work, we present a computational method for solving second kind nonlinear Fredholm Volterra integral equations which is based on the use of Haar wavelets. These functions together with the collocation method are then utilized to reduce the Fredholm Volterra integral equations to the solution of algebraic equations. Finally, we also give some numerical examples that shows validity and applicability of the technique.

*Key words:* Nonlinear Fredholm Volterra integral equation; Haar wavelet; Haar coefficient matrix; Block-Pulse Function; Collocation points.

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## 1 Introduction

Beginning from 1991 the wavelet method has been applied for solving integral equations, a short survey on this papers can be found in [5]. The solutions are often quite complicated and the advantages of the wavelet method get lost, therefore any kind of simplifications are welcome. One possibility for it is to make use of the Haar wavelets. In fact, Haar wavelets have a number of advantages, including: simplicity, orthogonality and very compact support. The main benefits of the Haar wavelets method are sparse representation, fast transformation and possibility of implementation of fast algorithm in matrix representation. The Haar basis is simplest instance of spline wavelets, resulting when the polynomial degree is set to zero, so computational costs with Haar wavelets is lesser. Different kind of basis functions have been used to solve and reduce integral equations to a system of algebraic equations [1-15]. The aim of this work is to present a numerical method for approximating the solution of nonlinear Fredholm Volterra integral equation of the second kind

$$f(x) = g(x) + \lambda_1 \int_0^x k_1(x,t)[f(t)]^m dt + \lambda_2 \int_0^1 k_2(x,t)[f(t)]^n dt, \quad (1.1)$$

where  $0 \leq x, t \leq 1$ ,  $m, n \geq 1$ ,  $g(x)$ ,  $k_1(x,t)$  and  $k_2(x,t)$  are assumed to be in  $L^2(R)$  on the interval  $0 \leq x, t < 1$ . We assume that Eq. (1.1) has a unique solution  $f$  to be determined.

**Definition 1.** The Haar wavelet is the function defined on the real line  $\mathbb{R}$  as:

$$H(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Now for  $n = 1, 2, \dots$ , write  $n = 2^j + k$  with  $j = 0, 1, \dots$  and  $k = 0, 1, \dots, 2^j - 1$  and define  $h_n(t) = 2^{\frac{j}{2}} H(2^j t - k)|_{[0,1]}$ . Also, define  $h_0(t) = 1$  for all  $t$ . Here the integer  $2^j$ ,  $j = 0, 1, \dots$ , indicates the level of the wavelet and  $k = 0, 1, \dots, 2^j - 1$  is the translation parameter. It can be shown that the sequence  $\{h_n\}_{n=0}^{\infty}$  is a complete orthonormal system in  $L^2[0, 1]$  and for  $f \in C[0, 1]$ ,

the series  $\sum_n \langle f, h_n \rangle h_n$  converges uniformly to  $f$  [17], where  $\langle f, h_n \rangle = \int_0^1 f(x)h_n(x)dx$ .

## 2 Materials and Methods

A function  $f(x)$  defined over the interval  $[0, 1)$  may be expanded as:

$$f(x) = \sum_{n=0}^{\infty} f_n h_n(x), \quad (2.1)$$

with  $f_n = \langle f(x), h_n(x) \rangle$ , that it is an inner product on the unit interval. In practice, only the first  $k$ -term of (2.1) are considered, where  $k$  is a power of 2, that is,

$$f(x) \simeq f_k(x) = \sum_{n=0}^{k-1} f_n h_n(x), \quad (2.2)$$

with matrix form:

$$f(x) \simeq f_k(x) = \mathbf{f}^t \mathbf{h}(x),$$

where,  $\mathbf{f} = [f_0, f_1, \dots, f_{k-1}]^t$  and  $\mathbf{h}(x) = [h_0(x), h_1(x), \dots, h_{k-1}(x)]^t$ . For a positive integer  $m$ ,  $[f(x)]^m$  may be approximated as:

$$[f(x)]^m \simeq \sum_{n=0}^{k-1} \tilde{f}_n h_n(x) = \tilde{\mathbf{f}}^t \mathbf{h}(x),$$

where  $\tilde{\mathbf{f}}$  is a column vector whose elements are nonlinear combinations of the elements of the vector  $\mathbf{f}$ . In the next section, we consider evaluation of  $\tilde{\mathbf{f}}$  in terms of  $\mathbf{f}$ .

Similarly,  $k(x, t) \in L^2[0, 1)^2$  may be approximated in the matrix form as

$$k(x, t) \simeq \mathbf{h}^t(x) \mathbf{k} \mathbf{h}(t),$$

where,  $\mathbf{k} = [k_{ij}]_{0 \leq i, j \leq k-1}$  and  $k_{ij} = \langle h_i(x), \langle k(x, t), h_j(t) \rangle \rangle$ , approximation of the kernel  $k(x, t)$  by wavelets is known as standard

representation. It is a wavelet image of the kernel and is usually a sparse matrix.

### 3 Evaluating $\tilde{\mathbf{f}}$

For numerical implementation of the proposed method, we need to calculate vector  $\tilde{\mathbf{f}}$  whose elements are nonlinear combination of the elements of the vector  $\mathbf{f}$ . For this purpose, we present the Haar coefficient matrix  $\mathbf{H}$ ; it is a  $k \times k$  matrix with the elements

$$\mathbf{H} = [h_n(t_j)]_{0 \leq n \leq k-1, 1 \leq j \leq k},$$

where the points  $t_j$  are the collocation points

$$t_j = \frac{j - \frac{1}{2}}{k}, \quad j = 1, 2, \dots, k.$$

Also, we define a  $k$ -set of Block-Pulse Function (BPF) as:

$$B_i(t) = \begin{cases} 1, & \frac{i-1}{k} \leq t < \frac{i}{k}, \text{ for all } i = 1, 2, \dots, k, \\ 0, & \text{elsewhere.} \end{cases} \quad (3.1)$$

The functions  $B_r(t)$  are disjoint and orthogonal. That is,

$$B_j(t)B_i(t) = \begin{cases} 0, & i \neq j, \\ B_i(t), & i = j, \end{cases} \quad (3.2)$$

$$\langle B_i(t), B_j(t) \rangle = \begin{cases} 0, & i \neq j, \\ \frac{1}{k}, & i = j. \end{cases} \quad (3.3)$$

It can be shown that  $\mathbf{h}(t) = \mathbf{HB}(t)$  [16], vector  $\mathbf{h}(t)$  and matrix  $\mathbf{H}$  are already introduced and  $\mathbf{B}(t) = [B_1(t), \dots, B_k(t)]^t$ . Using the subject already discussed in section 2,

$$f(x) = \mathbf{f}^t \mathbf{h}(x) \text{ and } [f(x)]^m = \tilde{\mathbf{f}}^t \mathbf{h}(x).$$

So,

$$\tilde{\mathbf{f}}^t \mathbf{h}(x) = [\mathbf{f}^t \mathbf{h}(x)]^m \quad (3.4)$$

orthonormality of the sequence  $\{h_n\}$  on  $[0, 1)$ , implies that

$$\int_0^1 \mathbf{h}(x)\mathbf{h}^t(x)dx = \mathbf{I}_{k \times k},$$

where,  $\mathbf{I}_{k \times k}$  is the identity matrix of order  $k$ , so, from (3.4) we have

$$\tilde{\mathbf{f}}^t = \int_0^1 \tilde{\mathbf{f}}^t \mathbf{h}(x)\mathbf{h}^t(x)dx = \int_0^1 [\mathbf{f}^t \mathbf{h}(x)]^m \mathbf{h}^t(x)dx.$$

Hence,

$$\begin{aligned} \tilde{\mathbf{f}}^t &= \int_0^1 [\mathbf{f}^t \mathbf{h}(x)]^m \mathbf{h}^t(x)dx \\ &= \int_0^1 [\mathbf{f}^t \mathbf{h}(x)]^{m-1} \mathbf{f}^t \mathbf{h}(x)\mathbf{h}^t(x)dx \\ &= \int_0^1 [\mathbf{f}^t \mathbf{H}\mathbf{B}(x)]^{m-1} \mathbf{f}^t \mathbf{H}\mathbf{B}(x)\mathbf{B}^t(x)\mathbf{H}^t dx. \end{aligned} \quad (3.5)$$

From (3.1) we have

$0 \leq t < \frac{1}{k}$  implies that  $B_1(t) = 1$  and  $B_i(t) = 0$  for  $i = 2, \dots, k$ .

$\frac{1}{k} \leq t < \frac{2}{k}$  implies that  $B_2(t) = 1$  and  $B_i(t) = 0$  for  $i = 1, \dots, k$  and  $i \neq 2$ .

$\vdots$

$\frac{k-1}{k} \leq t < 1$  implies that  $B_k(t) = 1$  and  $B_i(t) = 0$  for  $i = 1, \dots, k-1$ .

Also, disjoint property of BPFs leads to

$$\mathbf{B}(t)\mathbf{B}^t(t) = \begin{pmatrix} B_1(t) & & & O \\ & B_2(t) & & \\ & & \ddots & \\ O & & & B_k(t) \end{pmatrix}.$$

Now,  $x \in [\frac{i-1}{k}, \frac{i}{k})$  implies that  $\mathbf{B}(x) = \mathbf{e}_i$  where  $\mathbf{e}_i$  is  $i$ -th column of the identity matrix of order  $k$  so

$$\begin{aligned} \mathbf{HB}(x)\mathbf{B}^t(x)\mathbf{H}^t &= \begin{pmatrix} H_{0,1} & \dots & H_{0,k} \\ H_{1,1} & \dots & H_{1,k} \\ \vdots & \ddots & \vdots \\ H_{k-1,1} & \dots & H_{k-1,k} \end{pmatrix} \begin{pmatrix} 0 & & \emptyset \\ & \ddots & \\ & & 1 \\ \emptyset & & & \ddots \\ & & & & 0 \end{pmatrix} \\ &= \begin{pmatrix} H_{0,1} & \dots & H_{k-1,1} \\ H_{0,2} & \dots & H_{k-1,2} \\ \vdots & \ddots & \vdots \\ H_{0,k} & \dots & H_{k-1,k} \end{pmatrix} \\ &= \begin{pmatrix} H_{0,i}H_{0,i} & H_{0,i}H_{1,i} & \dots & H_{0,i}H_{k-1,i} \\ H_{1,i}H_{0,i} & H_{1,i}H_{1,i} & \dots & H_{1,i}H_{k-1,i} \\ \vdots & \vdots & \ddots & \vdots \\ H_{k-1,i}H_{0,i} & H_{k-1,i}H_{1,i} & \dots & H_{k-1,i}H_{k-1,i} \end{pmatrix}, \end{aligned}$$

hence,

$$\begin{aligned} \mathbf{f}^t\mathbf{HB}(x)\mathbf{B}^t(x)\mathbf{H}^t &= \left[ H_{0,i} \sum_{r=0}^{k-1} f_r H_{r,i}, H_{1,i} \sum_{r=0}^{k-1} f_r H_{r,i}, \dots, H_{k-1,i} \right. \\ &\quad \left. \sum_{r=0}^{k-1} f_r H_{r,i} \right]. \end{aligned} \tag{3.6}$$

Again for  $x \in [\frac{i-1}{k}, \frac{i}{k})$  we can write

$$\begin{aligned} \mathbf{f}^t \mathbf{H} \mathbf{B}(x) &= [f_0, f_1, \dots, f_{k-1}] \begin{pmatrix} H_{0,1} & H_{0,2} & \dots & H_{0,k} \\ H_{1,1} & H_{1,2} & \dots & H_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ H_{k-1,1} & H_{k-1,2} & \dots & H_{k-1,k} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \sum_{r=0}^{k-1} f_r H_{r,i}. \end{aligned} \quad (3.7)$$

Therefore, for evaluating  $\tilde{\mathbf{f}}$  and by substituting (3.6)-(3.7) into (3.5) we can proceed as follows

$$\begin{aligned} \tilde{\mathbf{f}}^t &= \int_0^1 [\mathbf{f}^t \mathbf{H} \mathbf{B}(x)]^{m-1} \mathbf{f}^t \mathbf{H} \mathbf{B}(x) \mathbf{B}^t(x) \mathbf{H}^t dx \\ &= \sum_{i=1}^k \int_{\frac{i-1}{k}}^{\frac{i}{k}} \left( \sum_{r=0}^{k-1} f_r H_{r,i} \right)^{m-1} [H_{0,i} \sum_{r=0}^{k-1} f_r H_{r,i}, H_{1,i} \sum_{r=0}^{k-1} f_r H_{r,i}, \dots, H_{k-1,i} \\ &\quad \sum_{r=0}^{k-1} f_r H_{r,i}] dx \\ &= \frac{1}{k} \sum_{i=1}^k \left( \sum_{r=0}^{k-1} f_r H_{r,i} \right)^{m-1} [H_{0,i} \sum_{r=0}^{k-1} f_r H_{r,i}, H_{1,i} \sum_{r=0}^{k-1} f_r H_{r,i}, \dots, \\ &\quad H_{k-1,i} \sum_{r=0}^{k-1} f_r H_{r,i}] dx \\ &= \frac{1}{k} \left[ \sum_{i=1}^k H_{0,i} \left( \sum_{r=0}^{k-1} f_r H_{r,i} \right)^m, \dots, \sum_{i=1}^k H_{k-1,i} \left( \sum_{r=0}^{k-1} f_r H_{r,i} \right)^m \right], \end{aligned}$$

if we apply the definition  $\mathbf{H} = [h_n(t_j)]_{0 \leq n \leq k-1, 1 \leq j \leq k}$ , we obtain

$$\tilde{\mathbf{f}} = \frac{1}{k} \left[ \sum_{i=1}^k h_0(t_i) \left( \sum_{r=0}^{k-1} f_r h_r(t_i) \right)^m, \dots, \sum_{i=1}^k h_{k-1}(t_i) \left( \sum_{r=0}^{k-1} f_r h_r(t_i) \right)^m \right]^t. \quad (3.8)$$

#### 4 Second kind nonlinear Volterra Fredholm integral equation

Now consider the nonlinear Fredholm Volterra integral equation of the second kind with nonlinear regular part:

$$f(x) = g(x) + \lambda_1 \int_0^x k_1(x, t)[f(t)]^m dt + \lambda_2 \int_0^1 k_2(x, t)[f(t)]^n dt, \quad (4.1)$$

where  $0 \leq x, t \leq 1$  and  $m, n \geq 1$  as before, in the matrix form we have:

$$f(x) \simeq \mathbf{h}^t(x)\mathbf{f}, \quad (4.2)$$

$$g(x) \simeq \mathbf{h}^t(x)\mathbf{g}, \quad (4.3)$$

$$k_1(x, t) \simeq \mathbf{h}^t(x)\mathbf{k}_1\mathbf{h}(t), \quad (4.4)$$

$$k_2(x, t) \simeq \mathbf{h}^t(x)\mathbf{k}_2\mathbf{h}(t), \quad (4.5)$$

$$[f(x)]^m \simeq \mathbf{h}^t(x)\tilde{\mathbf{f}}^m, \quad (4.6)$$

$$[f(x)]^n \simeq \mathbf{h}^t(x)\tilde{\mathbf{f}}^n, \quad (4.7)$$

by substituting the approximations (4.2)-(4.7) into (4.1) we obtain

$$\mathbf{h}^t(x)\mathbf{f} = \mathbf{h}^t(x)\mathbf{g} + \lambda_1 \int_0^x \mathbf{h}^t(x)\mathbf{k}_1\mathbf{h}(t)\mathbf{h}^t(t)\tilde{\mathbf{f}}^m dt \quad (4.8)$$

$$+ \lambda_2 \int_0^1 \mathbf{h}^t(x)\mathbf{k}_2\mathbf{h}(t)\mathbf{h}^t(t)\tilde{\mathbf{f}}^n dt$$

$$= \mathbf{h}^t(x)\mathbf{g} + \lambda_1 \mathbf{h}^t(x)\mathbf{k}_1 \left( \int_0^x \mathbf{h}(t)\mathbf{h}^t(t) dt \right) \tilde{\mathbf{f}}^m \quad (4.9)$$

$$+ \lambda_2 \mathbf{h}^t(x)\mathbf{k}_2 \left( \int_0^1 \mathbf{h}(t)\mathbf{h}^t(t) dt \right) \tilde{\mathbf{f}}^n$$

$$= \mathbf{h}^t(x)\mathbf{g} + \lambda_1 \mathbf{h}^t(x)\mathbf{k}_1\mathbf{s}(x)\tilde{\mathbf{f}}^m \quad (4.10)$$

$$+ \lambda_2 \mathbf{h}^t(x)\mathbf{k}_2\tilde{\mathbf{f}}^n, \quad (4.11)$$



where,  $\mathbf{s}(x) = \int_0^x \mathbf{h}(t)\mathbf{h}^t(t)dt$ . Now for evaluating  $s(x)$  at the collocation points  $t_j$  we may proceed as follows

$$\begin{aligned}
\mathbf{B}(x)\mathbf{B}^t(x) &= \begin{bmatrix} B_1(x) & \emptyset \\ & \ddots \\ \emptyset & B_k(x) \end{bmatrix} \\
&= B_1(x) \begin{bmatrix} 1 & \emptyset \\ & 0 \\ & & \ddots \\ \emptyset & & & 0 \end{bmatrix} + B_2(x) \begin{bmatrix} 0 & \emptyset \\ & 1 \\ & & 0 \\ & & & \ddots \\ \emptyset & & & & 0 \end{bmatrix} + \dots + B_k(x) \begin{bmatrix} 0 & \emptyset \\ & 0 \\ & & \ddots \\ & & & 0 \\ \emptyset & & & & 1 \end{bmatrix} \\
&= \sum_{i=1}^k B_i(x)\mathbf{d}^{(i)},
\end{aligned}$$

where,  $\mathbf{d}^{(i)}$  is a  $k \times k$  matrix with the elements

$$\mathbf{d}_{mn}^{(i)} = \begin{cases} 1, & m = n = i, \\ 0, & m \neq i \text{ or } n \neq i, \end{cases}$$

therefore we have

$$\begin{aligned}
\mathbf{h}(x)\mathbf{h}^t(x) &= \mathbf{H}\mathbf{B}(x)\mathbf{B}^t(x)\mathbf{H}^t \\
&= \mathbf{H}\left(\sum_{i=1}^k B_i(x)\mathbf{d}^{(i)}\right)\mathbf{H}^t \\
&= \sum_{i=1}^k B_i(x)\mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t. \tag{4.12}
\end{aligned}$$

By integrating (4.9) we obtain:

$$\begin{aligned}
\mathbf{s}(t) &= \int_0^t \mathbf{h}(x)\mathbf{h}^t(x)dx \\
&= \sum_{i=1}^k \int_0^t B_i(x)dx \mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t \\
&= \sum_{i=1}^k n_i(t)\mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t, \tag{4.13}
\end{aligned}$$

where,  $n_i(t) = \int_0^t B_i(x)dx$ ,  $t \in [0, 1]$ . Now by using (3.1) and simple calculation we obtain,

$$\begin{aligned}
n_1(t_1) &= \int_0^{t_1} B_1(x)dx = \frac{1}{2k} \text{ and } n_i(t_1) = 0 \text{ for } i = 2, \dots, k. \\
n_1(t_2) &= \int_0^{t_2} B_1(x)dx = \frac{1}{k}, n_2(t_2) = \int_0^{t_2} B_2(x)dx = \frac{1}{2k} \text{ and } n_i(t_2) = 0 \\
&\text{for } i = 3, \dots, k.
\end{aligned}$$

⋮

$$\begin{aligned}
n_1(t_k) &= \int_0^{t_k} B_1(x)dx = \frac{1}{k}, \dots, n_{k-1}(t_k) = \int_0^{t_k} B_{k-1}(x)dx = \frac{1}{k} \text{ and} \\
n_k(t_k) &= \int_0^{t_k} B_k(x)dx = \frac{1}{2k}.
\end{aligned}$$

So by evaluating (4.10) at the collocation points  $t_j$  we obtain

$$\begin{aligned}
\mathbf{s}(t_1) &= \frac{1}{2k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t, \\
\mathbf{s}(t_2) &= \frac{1}{k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t + \frac{1}{2k}\mathbf{H}\mathbf{d}^{(2)}\mathbf{H}^t, \\
&\vdots \\
\mathbf{s}(t_k) &= \frac{1}{k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t + \dots + \frac{1}{k}\mathbf{H}\mathbf{d}^{(k-1)}\mathbf{H}^t + \frac{1}{2k}\mathbf{H}\mathbf{d}^{(k)}\mathbf{H}^t,
\end{aligned}$$

or in abstract form

$$\begin{aligned}
\mathbf{s}(t_1) &= \frac{1}{2k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t, \\
\mathbf{s}(t_j) &= \frac{1}{k} \sum_{i=1}^{j-1} \mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t + \frac{1}{2k}\mathbf{H}\mathbf{d}^{(j)}\mathbf{H}^t, \quad \text{for } j = 2, \dots, k.
\end{aligned}$$

Collocating (4.8) at the points  $t_j, j = 1, 2, \dots, k$  gives

$$\mathbf{h}^t(t_j)\mathbf{f} = \mathbf{h}^t(t_j)\mathbf{g} + \lambda_1\mathbf{h}^t(t_j)\mathbf{k}_1\mathbf{s}(t_j)\tilde{\mathbf{f}}^m + \lambda_2\mathbf{h}^t(t_j)\mathbf{k}_2\tilde{\mathbf{f}}^n, \tag{4.14}$$

which is a nonlinear system of algebraic equations, and can be solved for elements  $f_0, f_1, \dots, f_{k-1}$  by Newton's iterative method and desired approximation for  $f(x)$  can be obtained by  $f_k(x)$  as

$$f_k(x) = \sum_{n=0}^{k-1} f_n h_n(x).$$

## 5 Error Analysis

**Theorem 1.** If a differentiable function  $f(x)$  with bounded first derivative on  $(0,1)$  is represented in a series of Haar wavelets we have  $\|f_k(x) - f(x)\| \leq \frac{M}{\sqrt{3}^k}$ , which implies that  $\lim_{k \rightarrow \infty} f_k(x) = f(x)$ .

In [1] it is established that if  $x_i \in [0, 1), i = 1, \dots, l$  be  $l$  equidistance points and calculate  $f'(x_i)$  for  $i = 1, 2, \dots, l$ , then  $\varepsilon + \max_{1 \leq i \leq l} |f'(x_i)|$  may be considered as an estimation of  $M$ . Clearly, the estimation would become more precise if  $l$  increases and  $\varepsilon$  can be chosen by user (say,  $\varepsilon = 1$ ).

**Proof.** See [1].

## 6 Numerical Examples

Now for numerical implementing of presented method, we choose 2 examples with exact solution for comparing with the approximate solution. Results have shown in table 1 and table 2 for example 1 and example 2 respectively for  $k=8$ .

**Example 1.** Consider the following nonlinear Volterra-Fredholm integral equation

$$f(x) = e^x - \frac{1}{2}(e^{2x} - 1) + \int_0^x [f(t)]^2 dt, \quad 0 \leq x < 1,$$

with exact solution  $f(x) = e^x$ .

Table 1: Numerical result for example 1 with  $k=8$

t	Exact	Approximate for k=8
0.1	1.1051	1.0684
0.2	1.2214	1.2121
0.3	1.3498	1.3758
0.4	1.4918	1.5529
0.5	1.6487	1.7080
0.6	1.8221	1.7780
0.7	2.0137	2.0278
0.8	2.2255	2.2744
0.9	2.4596	2.5322

**Example 2.** Consider the following nonlinear Volterra-Fredholm integral equation

$$f(x) = e^x - \frac{1}{9}(1 + 2e^3)x + \int_0^1 xt[f(t)]^3 dt, \quad 0 \leq x < 1,$$

with exact solution  $f(x) = e^x$ .

Table 2: Numerical result for example 2 with k=8

t	Exact	Approximate for k=8
0.1	1.1051	1.0658
0.2	1.2214	1.2091
0.3	1.3498	1.3712
0.4	1.4918	1.5547
0.5	1.6487	1.7225
0.6	1.8221	1.7625
0.7	2.0137	1.9978
0.8	2.2255	2.2641
0.9	2.4596	2.5258

## 7 Conclusion

In presented work we introduced a method to solve the nonlinear Volterra Fredholm integral equations. Haar wavelets together with the collocation method were used to reduce the problem to the solution of nonlinear algebraic equations. For other orthogonal polynomials such as Legendre and Chebyshev polynomials the calculation procedures are usually too tedious, although some recursive formula are available. These polynomials are in no way able to compare with Haar wavelets expansion with respect to computation time and data storage requirements. So the fast, local and multiplicative properties of Haar wavelets were used for solving second kind nonlinear Fredholm Volterra integral equations. Error analysis states more accurate of the approximated solution may be obtained by using larger  $k$ .

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