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Application of variational iteration method for solving singular two point boundary value problems

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Abstract

DEA methodology allows DMUs to select the weights freely, so in the optimal solution we may see many zeros in the optimal weight. to overcome this problem, there are some methods, but they are not suitable for evaluating DMUs with fuzzy data. In this paper, we propose a new method for solving fuzzy DEA models with restricted multipliers with less computation, and compare this method with Liu'[11]. Finally, by the proposed method, we evaluate a flexible manufacturing system with little computation, and then we compared the computational complexity of our proposed method with that of liu's method.

Key words: DEA, Fuzzy DEA, Cone Ratio, Assurance Region.

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1 Introduction

Data Envelopment Analysis (DEA) is nonparametric technique for evaluating the relative efficiency of decision making units (DMUs) with common inputs and outputs.

This technique was initially proposed by Charnes et al.[2] and was improved by others [1,4].
Recently, the use of data envelopment analysis has been recommended as a discrete alternative multiple criteria tool for evaluation of manufacturing technologies and flexible manufacturing systems (FMSs). Khouja [8] introduced a two-phase approach for the selection of an advanced manufacturing technology from a set of feasible technology alternatives.

DEA methodology allows DMUs to select the weights that are most favorable, so in the optimal solution we may see may zeros in the optimal weight. For resolving of this problem, Thompson et al.[12] proposed the concept of the assurance region(AR) to restrict the ratio of any two weighs. And then charnes et al. [2] improved the approach and called it the Cone-Ratio (CR) method.

In traditional DEA models, we assume that all inputs and outputs are exactly known. But in real world, this assumption may not always be true.

On the other hand, in more general cases, the data for evaluation are stated by natural language such as good, bad, to reflect the general situation. Mathematical models have been developed to quantify performance measures such as quality and flexibility for justifying investment in advanced manufacturing systems. Kahraman [7] developed a fuzzy hierarchical TOPSIS model for the multi-criteria evaluation of industrial robotic systems. Also, some researchers have proposed DEA fuzzy models to evaluate DMUs with fuzzy data [5,6,10].

However, methods of restriction of multipliers proposed in DEA are not suitable for evaluating DMUs with fuzzy data. So Liu [11] expanded assurance region method for DMUs with fuzzy data.

In this paper, we introduced the fuzzy number in section 2, and then proposed a new method for the fuzzy AR model in section 3, we introduced the fuzzy CR model in section 4, and provide a numerical example and evaluate fuzzy FMSs and compare the proposed method in this paper with Liu'[11].

2 Fuzzy numbers

A fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

$\mu_A(x)$ is called the membership function of x in A.
A fuzzy number M is a convex normalized fuzzy set M of real line R such that :
1) there exists exactly one $x \in R$ with $\mu_M(x) = 1$.
2) $\mu_M(x)$ is piecewise continuous.
The (crisp) set of element that belong to the fuzzy set A at least to the degree α-cut set:
$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$.
The lower and upper end points of any α-cut set, A_α, are represented by $[\mathrm{A}]^l_\alpha$ and $[\mathrm{A}]^u_\alpha$, respectively.
For arbitrary fuzzy numbers
$D(v, u) = \int_0^1 S(\alpha)([u]^l_\alpha + [u]^u_\alpha - [v]^l_\alpha - [v]^u_\alpha)d\alpha$
is the distance between u,v and $S(\alpha)$ is an increasing function and $\int_0^1 S(\alpha) = \frac{1}{2}$. [9,13,14]
Now, we define the ranking system on F(R) as:
1) $u \succ v$ if $D(v, u) > 0$,
2) $u \prec v \ if \ D(v, u) < 0$,
3) $u \cong v \ if \ D(v, u) = 0$.
Definition 1. Let $\tilde{f} : \ X \longrightarrow F(R)$ be a fuzzy function from $X \subset F(R)$ into $F(R)$. The fuzzy number $\widetilde{UB}$ is a distance based upper bound for $\tilde{f}$ if $D(\widetilde{UB}, \widetilde{A}) \geq 0, \ \forall \widetilde{A}, \ \widetilde{A} \in \tilde{f}$.

3 Fuzzy AR model

Now we consider the AR model [12].

$$\begin{aligned}
\max\ & \sum_{r=1}^{s} u_r y_{ro} && (1)\\
s.t.\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, && j = 1, ..., n, \\
& -v_p + C_{pq}^{l} v_q \leq 0\ ,\ v_p - C_{pq}^{u} v_q \leq 0, && \forall p;\ p < q = 2, ..., m, \\
& -u_a + D_{ab}^{l} u_b \leq 0\ ,\ u_a - D_{ab}^{u} u_b \leq 0, && \forall a; a < b = 2, ..., s, \\
& u_r \geq 0, && r = 1, ..., s, \\
& v_i \geq 0 && i = 1, ..., m.
\end{aligned}$$

Let us assume that we have a set of DMUs with fuzzy input-output vectors $(\tilde{x}_j, \tilde{y}_j)$, in which $\tilde{x}_j \in F(R)_{\geq 0}$ and $\tilde{y}_j \in F(R)_{\geq 0}$ where $F(R)_{\geq 0}$ is the family of all nonnegative fuzzy numbers.
Now we consider the AR model and extended this model to the fuzzy case (FAR):

$$\begin{aligned}
\max\ & \sum_{r=1}^{s} u_r \tilde{y}_{ro} && (2)\\
s.t.\ & \sum_{i=1}^{m} v_i \tilde{x}_{io} \cong \tilde{1}, \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \preceq \tilde{0}, && j = 1, ..., n, \\
& -v_p + C_{pq}^{l} v_q \leq 0\ ,\ v_p - C_{pq}^{u} v_q \leq 0, && \forall p;\ p < q = 2, ..., m, \\
& -u_a + D_{ab}^{l} u_b \leq 0\ ,\ u_a - D_{ab}^{u} u_b \leq 0, && \forall a;\ a < b = 2, ..., s, \\
& u_r \geq 0, && r = 1, ..., s, \\
& v_i \geq 0 && i = 1, ..., m.
\end{aligned}$$

We suppose DMU_o is the DMU under assessment. We seek a suitable upper bound, $\widetilde{UB}$, for the objective function; therefore, we must minimize $d(\widetilde{UB},\sum_{r=1}^{s} u_r\tilde{y}_{ro})$ under feasibility. So we have:

$$
\begin{aligned}
z = \min \quad & \textstyle\int_0^1 S(\alpha)([\widetilde{UB}]_\alpha^u + [\widetilde{UB}]_\alpha^l - \sum_{r=1}^{s} u_r([\tilde{y}_{ro}]_\alpha^u + [\tilde{y}_{ro}]_\alpha^l))d\alpha \qquad (3)\\
s.t \quad & \textstyle\int_0^1 S(\alpha)(\sum_{i=1}^{m} v_i([\tilde{x}_{io}]_\alpha^u + [\tilde{x}_{io}]_\alpha^l) - (1+1))d\alpha = o,\\
& \textstyle\int_0^1 S(\alpha)(\sum_{r=1}^{s} u_r([\tilde{y}_{rj}]_\alpha^u + [\tilde{y}_{rj}]_\alpha^l) - \sum_{i=1}^{m} v_i([\tilde{x}_{ij}]_\alpha^u + [\tilde{x}_{ij}]_\alpha^l))d\alpha \leq 0,\ j = 1,...,n,\\
& -v_p + C_{pq}^l v_q \leq 0\ ,\ v_p - C_{pq}^u v_q \leq 0,\\
\forall p;\ p < q = 2,...,m & \\
& -u_a + D_{ab}^l u_b \leq 0\ ,\ u_a - D_{ab}^u u_b \leq 0,\\
\forall a;\ a < b = 2,...,s & \\
& u_r \geq 0, \quad r = 1,...,s\\
& v_i \geq 0, \quad i = 1,...,m.
\end{aligned}
$$

Theorem 1: $\tilde{1}$ is an upper bound for $\tilde{f}: U_0 \longrightarrow F(R)$ where $\tilde{f}(\alpha,\beta) = \sum_{r=1}^{s} u_r\tilde{y}_{ro}$ and $U_0 \subseteq R^k \times R^l$ and $o \in \{1,2,...,n\}$, where U_0 is the feasible set of model(2), corresponding to DMU_o.
Proof: considering DMU_o is under assessment, we have:

$$\int_0^1 S(\alpha)(\sum_{r=1}^{s} u_r([\tilde{y}_{rj}]_\alpha^u+[\tilde{y}_{rj}]_\alpha^l))d\alpha \leq \int_0^1 S(\alpha)(\sum_{i=1}^{m} v_i([\tilde{x}_{ij}]_\alpha^u+[\tilde{x}_{ij}]_\alpha^l))d\alpha,$$

and

$$\int_0^1 S(\alpha)(\sum_{i=1}^{m} v_i([\tilde{x}_{io}]_\alpha^u + [\tilde{x}_{io}]_\alpha^l)) = \int_0^1 2S(\alpha)$$

and therefore

$$
\begin{aligned}
D(\tilde{1},\sum_{r=1}^{s} u_{ro}\tilde{y}_{ro}) &= \textstyle\int_0^1 2S(\alpha)d\alpha - \int_0^1 S(\alpha)(\sum_{r=1}^{s} u_r([\tilde{y}_{ro}]_\alpha^u\\
&\quad + [\tilde{y}_{ro}]_\alpha^l))d\alpha \geq \textstyle\int_0^1 2S(\alpha)d\alpha\\
&\quad - \textstyle\int_0^1 2S(\alpha)d\alpha = 0
\end{aligned}
$$

For simplification, we convert model 3 to model4 by using the following variable changes:

$$\hat{x}_{ij} = \int_0^1 S(\alpha)([\tilde{x}_{ij}]^u_\alpha + [\tilde{x}_{ij}]^l_\alpha)d\alpha$$

$$\hat{y}_{rj} = \int_0^1 S(\alpha)([\tilde{y}_{rj}]^u_\alpha + [\tilde{y}_{rj}]^l_\alpha)d\alpha$$

$$U = \int_0^1 S(\alpha)([\widetilde{UB}]^u_\alpha + [\widetilde{UB}]^l_\alpha)d\alpha$$

Since $S(\alpha)$ is an arbitrary increasing function, we can replace $S(\alpha)$ by α, for computational simplicity. It is evident that:

$$-z + U = \max \int_0^1 S(\alpha) \sum_{r=1}^{s} u_i([\tilde{y}_{ro}]^u_\alpha + [\tilde{y}_{ro}]^l_\alpha)d\alpha.$$

Therefore model (3) reduces to the following model:

$$\begin{array}{lll} -z + U = max & \sum_{r=1}^{s} u_r \hat{y}_{ro} & \\ s.t. & \sum_{i=1}^{m} v_i \hat{x}_{io} = 1, & \\ & \sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \leq 0, & j = 1, ..., n, \\ & -v_p + C^l_{pq} v_q \leq 0 \ , \ v_p - C^u_{pq} v_q \leq 0, & \forall p;\ p < q = 2, ..., m, \\ & -u_a + D^l_{ab} u_b \leq 0 \ , \ u_a - D^u_{ab} u_b \leq 0, & \forall a;\ a < b = 2, ..., s, \\ & u_r \geq 0, & r = 1, ..., s, \\ & v_i \geq 0, & i = 1, ..., m. \end{array} \quad (4)$$

4 Fuzzy CR model

Consider the CR model [2].

$$z = max \sum_{r=1}^{s}\sum_{q=1}^{l} \beta_q b_{qr} y_{ro} \tag{5}$$

$$\begin{aligned} s.t. \quad & \\ & \sum_{i=1}^{m}\sum_{p=1}^{k} \alpha_p a_{pi} x_{io} = 1, \\ & \sum_{r=1}^{s}\sum_{q=1}^{l} \beta_q b_{qr} y_{rj} - \sum_{i=1}^{m}\sum_{p=1}^{k} \alpha_p a_{pi} x_{ij} \leq 0, \; j = 1, ..., n, \\ & \alpha_p \geq 0, \qquad p = 1, ..., k, \\ & \beta_q \geq 0, \qquad q = 1, ..., l. \end{aligned}$$

Now we extended the CR model to the fuzzy case (FCR) with fuzzy parameters :

$$z = max \sum_{r=1}^{s}\sum_{q=1}^{l} \beta_q b_{qr} \tilde{y}_{ro} \tag{6}$$

$$\begin{aligned} s.t \quad & \sum_{i=1}^{m}\sum_{p=1}^{k} \alpha_p a_{pi} \tilde{x}_{io} \cong \tilde{1}, \\ & \sum_{r=1}^{s}\sum_{q=1}^{l} \beta_q b_{qr} \tilde{y}_{rj} - \sum_{i=1}^{m}\sum_{p=1}^{k} \alpha_p a_{pi} \tilde{x}_{ij} \preceq \tilde{0}, \; j = 1, ..., n, \\ & \alpha_p \geq 0, \qquad p = 1, ..., k, \\ & \beta_q \geq 0, \qquad q = 1, ..., l. \end{aligned}$$

Theorem 2: $\tilde{1}$ is an upper bound for $\tilde{f}: \; U_0 \longrightarrow F(R)$, where $\tilde{f}(\alpha, \beta) = \sum_{r=1}^{s}\sum_{q=1}^{l} \beta_q b_{qr} \tilde{y}_{ro} \; U_0 \subseteq R^k \times R^l$, and $o \in \{1, 2, ..., n\}$, where U_0 is the feasible set of model(6), corresponding to DMU_o .

Proof:The proof is similar to the proof of theorem 1, and is hence omitted.

By using the proposed method for the FAR model and theorem 2, we can easily transform the FAR model to a crisp model as follows:

$$
\begin{aligned}
-z+U = max \ & \sum_{r=1}^{s}\sum_{q=1}^{l}\beta_q b_{qr}\hat{y}_{ro} \\
s.t. \quad & \sum_{i=1}^{m}\sum_{p=1}^{k}\alpha_p a_{pi}\hat{x}_{io} = 1, \\
& \sum_{r=1}^{s}\sum_{q=1}^{l}\beta_q b_{qr}\hat{y}_{rj} - \sum_{i=1}^{m}\sum_{p=1}^{k}\alpha_p a_{pi}\hat{x}_{ij} \leq 0, \ j = 1, ..., n, \\
& \alpha_p \geq 0, \qquad p = 1, ..., k, \\
& \beta_q \geq 0, \qquad q = 1, ..., l.
\end{aligned}
$$

5 Numerical example

In this section, to illustrate the use of the methodology developed here, a numerical example is considered.
We have 12 FMSs with 2 inputs and 4 outputs, the data of which are summarized in table 1 and table 2. Input1 is a fuzzy number and input 2 is a crisp number, output 1 is also a crisp number but other outputs are fuzzy numbers. We know that a crisp number can be shown as a fuzzy number; therefore, we can solve this example by our proposed method.

FMSs	Capital and Operating cost	Floor space needed
1	(16.17,17.02,17.87)	5
2	(15.64,16.46,17.28)	4.5
3	(11.17,11.76,12.35)	6
4	(9.99,10.25,11.05)	4
5	(9.03,9.50,9.98)	3.8
6	(4.55,4.79,5.03)	5.4
7	(5.90,6.21,6.25)	6.2
8	(10.56,11.12,11.68)	6
9	(3.49,3.67,3.85)	8
10	(8.48,8.93,9.38)	7
11	(16.85,17.74,18.63)	7.1
12	(14.11,14.85,15.59)	6.2

Table 1. Inputs of DMUs

FMSAs	Qualitative	WIP(10)	NO. of Trady	Yield(00)
1	42	(43.045.347.6)	(13.5,14.2,14,9)	(28.6,30.1,31.6)
2	39	(38.1,40.1,42.1)	(12.4,13.0,13.7)	(28.3,29.8,31.3)
3	26	(37.6,39.6,41.6)	(13.1,13.8,14.5)	(23.3,24.5,25.7)
4	22	(34.2,36.0,37.8)	(10.7,11.3,11.9)	(23.8,25.0,26.3)
5	21	(32.5,34.2,35.9)	(11.4,12.0,12.6)	(19.4,20.4,21.4)
6	10	(19.1,20.1,21.1)	(4.8,5.0,5.3)	(15.7,16.5,17.3)
7	14	(25.2,26.5,27.8)	(6.7,7.0,7.4)	(18.7,19.7,20.7)
8	25	(34.13,5.9,37.7)	(8.6,9.0,9.5)	(23.5,24.7,25.9)
9	4	(16.5,17.4,18.3)	(0.1,0.10.1)	(17.2,18.1,19.0)
10	16	(32.6,34.3,36.0)	(6.2,6.5,6.8)	(19.6,20.6,21.6)
11	43	(43.3,45.6,47.9)	(13.3,14.0,14.7)	(29.5,31.1,32.7)
12	27	(36.8,38.7,40.6)	(13.1,13.8,14.5)	(24.1,25.4,26.7)

Table 2. Outputs of DMUs

By the proposed method in this paper, we have evaluated FMSs and the results of the evaluation are summarized in table 3.

FMSs	efficiency
1	0.9733
2	0.9560
3	0.9530
4	0.9953
5	1
6	0.9795
7	1
8	0.9390
9	0.8343
10	0.8303
11	0.9328
12	0.7909

Table 3. Result of the evaluation

By our method, we have evaluated each FMS by solving one LP. We can compare their efficiencies directly in our method, but for evaluating each FMS using Liu's method, we must solve the several models for different α-levels, and also in each α-level we should solve two LPs.
For example, in [11] each FMS has been evaluated at 11 distinct α-levels; therefore one must solve $2 \times 11 = 22$ LPs for evaluating each FMS. And also, after solving multiple models, we cannot compare these results directly and we need to use the ranking of fuzzy numbers.
Hence, using Liu's method leads to increasing the computational complexity, which is not reasonable.

6 Conclusion

In this paper, we introduced a new method for solving the fuzzy CR and AR models to evaluate flexible manufacturing systems. Although there are several methods for evaluating FMSs, they are not suitable for large problem in fuzzy FMSs, because they require a lot of computation. In this paper, we proposed a new method for evaluating FMSs with fuzzy data with little computation.

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