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New solution of fuzzy linear matrix equations

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Abstract

In this paper, a new method based on parametric form for approximate solution of fuzzy linear matrix equations (FLMEs) of the form $AX = B$, where A is a crisp matrix, B is a fuzzy number matrix and the unknown matrix X one, is presented. Then a numerical example is presented to illustrate the proposed model.

Key words: Fuzzy number; Parametric form; Fuzzy linear matrix equations

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1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [29], Dubois and Prade [14]. We refer the reader to [21] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems ranging from fuzzy metric spaces [27], fuzzy differential equations [5], fuzzy linear systems [3,4,11] and particle physics [15,16].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear system (FLS) [7,8,9,25,26]. Friedman *et al.* [18] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in FLS $Ax = Bx + y$ where A, B are real $n \times n$ matrices, the unknown vector x is a vector consisting of n fuzzy numbers and the constant y is a vector consisting of n fuzzy numbers [19]. In [1,3,4,11] the authors presented conjugate gradient, LU decomposition method for solving general FLS or symmetric FLS. Also, Abbasbandy *et al.* [6] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form $Ax + f = Bx + c$, where $A,\ B$ are real $m \times n$ matrices, the unknown vector x is a vector consisting of n fuzzy numbers and the constant $f,\ c$ are the vectors consisting of m fuzzy numbers. Then Allahviranloo *et al.* [10] introduced a general model for solving non-square FLS.

Recently, Ezzati [17] proposed a new method for solving an $n \times n$ FLS whose coefficients matrix was crisp and the right-hand side column was an arbitrary fuzzy number vector by using the embedding method given in Cong-Xin and Min [12] and replaced the original $n \times n$ fuzzy linear system by two $n \times n$ crisp linear systems. It is clear that in large systems, solving an $n \times n$ linear system is better than solving a $2n \times 2n$ linear system. Since perturbation analysis is very important in numerical methods, the authors of Wang *et al.* [28] presented the perturbation analysis for a class of fuzzy linear systems which could be solved by an embedding method. Now, according to the presented method in this paper, we can investigate perturbation analysis in two $n \times n$ crisp linear systems.

The paper is organized as follows: In Sect. 2, we introduce the notation, the definitions and preliminary results that will be used throughout the paper. In Section 3, we introduce FLMEs and the model for solving FLMEs is proposed. Then, we review the proposed method in Friedman *et al.* [18] and Ezzati [17] for solving fuzzy linear system. The proposed model is illustrated by solving an example in section 4 and conclusions are drawn in section 5.

2 Preliminaries

Parametric form of an arbitrary fuzzy number is given in [18,24] as follows.

Definition 2.1 *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r),\ 0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,*
2. *$\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$,*
3. *$\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1$.*

The set of all these fuzzy numbers is denoted by E which is a complete metric space with Hausdorff distance. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \leq r \leq 1$.
For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r)),\ y = (\underline{y}(r), \overline{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [24]

(a) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,
(b) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(c) $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Remark 2.1 *[2] Let $u = (\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$ be a fuzzy number, we take*

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2},$$

$$u^d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}.$$

It is clear that $u^d(r) \geq 0$, $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$, also a fuzzy number $u \in E^1$ is said symmetric if $u^c(r)$ is independent of r for all $0 \leq r \leq 1$.

Remark 2.2 *Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and also k, s are arbitrary real numbers. If $w = ku + sv$ then*

$$w^c(r) = ku^c(r) + sv^c(r),$$

$$w^d(r) = |k|u^d(r) + |s|v^d(r).$$

3 Fuzzy linear matrix equations

Definition 3.1 *A matrix system such as*

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & ea_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & \ldots & y_{1n} \\ y_{21} & y_{22} & \ldots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \ldots & y_{nn} \end{pmatrix}, \tag{3.1}$$

where a_{ik}, $1 \leq i, k \leq n$, are real numbers, the elements y_{ij}, $1 \leq i, j \leq n$ in the right-hand matrix are fuzzy numbers and the unknown elements x_{kj}, $1 \leq k, j \leq n$ are ones, is called a FLMEs.

Using matrix notation, we have

$$AX = Y. \tag{3.2}$$

A fuzzy number matrix

$$X = \Big(x_1, x_2, \ldots, x_n \Big),$$

given by $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj})^T,\ 1 \le j \le n$, is called a solution of the FLMEs (3.2) if

$$Ax_j = y_j, \quad 1 \le j \le n, \tag{3.3}$$

where $y_j = (y_{1j}, y_{2j}, \ldots, y_{nj})^T$ is the jth column of fuzzy matrix B.

Consider fuzzy linear system Eq.(3.1). By referring to Remark (2.2) we have linear matrix equations (LMEs)

$$\begin{cases} AX^c(r) = Y^c(r), \\ DX^d(r) = Y^d(r), \end{cases} \tag{3.4}$$

where $X^c(r) = (x_1^c(r), x_2^c(r), \ldots, x_n^c(r))^T$,
$X^d(r) = (x_1^d(r), x_2^d(r), \ldots, x_n^d(r))^T$,
$Y^c(r) = (y_1^c(r), y_2^c(r), \ldots, y_n^c(r))^T$ and
$Y^d(r) = (y_1^d(r), y_2^d(r), \ldots, y_n^d(r))^T$ and D contains the absolute values of A. We known that if A and D are nonsingular matrices then

$$\begin{cases} X^c(r) = A^{-1}Y^c(r), \\ X^d(r) = D^{-1}Y^d(r). \end{cases} \tag{3.5}$$

Therefore, we can solve FLS (3.1) by solving LMEs (3.4) and we have

$$\begin{aligned} \underline{X}(r) &= X^c(r) - X^d(r), \\ \overline{X}(r) &= X^c(r) + X^d(r). \end{aligned} \tag{3.6}$$

Theorem 3.1 *[13] A square crisp matrix is inverse-nonnegative if and only if it is the product of a permutation matrix by a diagonal* *matrix. And a square crisp matrix is inverse-nonnegative if and only if its entries are all zero except for a single positive entry in each row and column.*

Finally, we conclude this section by a reviewing on the proposed method for solving FLS in [18]. Consider FLMEs (3.1) and let X and Y are fuzzy number vector therefore, we have

$$Ax = y \tag{3.7}$$

where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.

Friedman *et al.* [18] wrote the FLS (3.7) as follows:

$$SX' = Y', \tag{3.8}$$

where s_{ij} are determined as follows:

$$\begin{aligned} a_{ij} \geq 0 &\Longrightarrow s_{ij} = a_{ij}, \; s_{i+m,j+n} = a_{ij}, \\ a_{ij} < 0 &\Longrightarrow s_{i,j+n} = -a_{ij}, \; s_{i+m,j} = -a_{ij}, \end{aligned} \tag{3.9}$$

and any s_{ij} which is not determined by (3.9) is zero and

$$X' = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \quad Y' = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_m \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_m \end{bmatrix}.$$

The structure of S implies that $s_{ij} \geq 0, 1 \leq i \leq 2m, 1 \leq j \leq 2n$ and

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \tag{3.10}$$

where B contains the positive entries of A, and C contains the absolute values of the negative entries of A, i.e., $A = B - C$.

Theorem 3.2 *[18] The matrix S is a nonsingular matrix if and only if the matrices $A = B - C$ and $B + C$ are both nonsingular matrices.*

Theorem 3.3 *[18] If S^{-1} exists it must have the same structure as S, i.e.*

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{3.11}$$

where

$$D = \frac{1}{2}[(B+C)^{-1} + (B-C)^{-1}], \qquad E = \frac{1}{2}[(B+C)^{-1} - (B-C)^{-1}].$$

We know that if S is a nonsingular matrix then

$$X' = S^{-1}Y'. \tag{3.12}$$

Recently, Ezzati [17] considered FLS (3.7) and solved by using the embedding approach. Ezzati [17] wrote the FLS (3.7) as follows:

$$A(\underline{x} + \overline{x}) = \underline{y} + \overline{y}, \tag{3.13}$$

where $h = (\underline{x} + \overline{x}) = (\underline{x}_1 + \overline{x}_1, \underline{x}_2 + \overline{x}_2, \ldots, \underline{x}_n + \overline{x}_n)^T$ and $\underline{y} + \overline{y} = (\underline{y}_1 + \overline{y}_1, \underline{y}_2 + \overline{y}_2, \ldots, \underline{y}_n + \overline{y}_n)^T$.

Theorem 3.4 *[17] Suppose the inverse of matrix A in Eq.(3.7) exists and $x = (x_1, x_2, \ldots, x_n)^T$ is a fuzzy solution of this equation. Then $\underline{x}(r) + \overline{x}(r)$ is the solution of the following system*

$$A(\underline{x}(r) + \overline{x}(r)) = \underline{y}(r) + \overline{y}(r). \tag{3.14}$$

We know that if A is a nonsingular matrix then

$$h = A^{-1}(\underline{y}(r) + \overline{y}(r)).$$

Let matrices B and C have defined as Eq.(3.10). Now using matrix notation for Eq.(3.1), we get

$$\begin{cases} B\underline{x}(r) - C\overline{x}(r) = \underline{y}(r), \\ B\overline{x}(r) - C\underline{x}(r) = \overline{y}(r). \end{cases}$$

By substituting of $\overline{x}(r) = h - \underline{x}(r)$ and $\underline{x}(r) = h - \overline{x}(r)$ in the first and second equation of above system, respectively, we have

$$(B + C)\underline{x}(r) = \underline{y}(r) + Ch \tag{3.15}$$

and

$$(B + C)\overline{x}(r) = \overline{y}(r) + Ch.$$

If $B + C$ is a nonsingular matrix then

$$\underline{x}(r) = (B + C)^{-1}(\underline{y}(r) + Ch).$$

Therefore, we can solve FLS (3.7) by solving Eqs. (3.14)-(3.15).

Theorem 3.5 *Assume that F_n, E_n and O_n are the number of multiplication operations that are required to calculate $X' = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, -\overline{x}_1, -\overline{x}_2, \ldots, -\overline{x}_n)^T = S^{-1}Y$ (the proposed method in Friedman et al. [18]), $X'' = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ from Eqs. (3.14)-(3.15) (the proposed method in Ezzati [17]) and $X'' = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ from Eqs. (3.5) and Eqs. (3.6), respectively. Then $O_n \leq E_n \leq F_n$ and $F_n - E_n = E_n - O_n = n^2$.*

Proof. We have

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix},$$

where

$$D = \frac{1}{2}[(B+C)^{-1} + (B-C)^{-1}], \qquad E = \frac{1}{2}[(B+C)^{-1} - (B-C)^{-1}].$$

Therefore, for determining S^{-1}, we need to compute $(B+C)^{-1}$ and $(B-C)^{-1}$. Now, assume that M is a $n \times n$ matrix and denote by

$h_n(M)$ the number of multiplication operations that are required to calculate M^{-1}. It is clear that

$$h(S) = h(B + C) + h(B - C) = 2h_n(A)$$

and hence

$$F_n = 2h_n(A) + 4n^2.$$

For computing $\underline{x}+\overline{x} = (\underline{x}_1+\overline{x}_1, \underline{x}_2+\overline{x}_2, \ldots, \underline{x}_n+\overline{x}_n)^T$ from Eq.(3.14) and $\underline{x} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)^T$ from Eq.(3.15), the number of multiplication operations are $h_n(A) + n^2$ and $h_n(B + C) + 2n^2$, respectively. Clearly $h_n(B + C) = h_n(A)$, so

$$E_n = 2h_n(A) + 3n^2$$

and hence $E_n - F_n = n^2$. For computing $X'' = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$, from Eqs. (3.5) and Eqs. (3.6), the number of multiplication operations are $h_n(A) + n^2$ and $h_n(D) + n^2$. Therefore

$$O_n = 2h_n(A) + 2n^2$$

and hence $F_n - E_n = E_n - O_n = n^2$. This proves theorem. $\square$

4 Numerical example

Example 4.1 *Consider the following FLMEs*

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} (-3 + 2r, 2 - 3r) & (-1 + 2r, 4 - 3r) \\ (7r, 11 - 4r) & (1 + 4r, 10 - 5r) \end{bmatrix}$$

By using Eqs. (3.4), we have:

$$\begin{bmatrix} x^c_{11}(r) & x^c_{12}(r) \\ x^c_{21}(r) & x^c_{22}(r) \end{bmatrix} = \begin{bmatrix} 1 & \frac{5}{2} - \frac{r}{2} \\ \frac{3}{2} - \frac{r}{2} & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} x_{11}^d(r) & x_{12}^d(r) \\ x_{21}^d(r) & x_{22}^d(r) \end{bmatrix} = \begin{bmatrix} 1-r & \frac{3}{2}-\frac{3}{2}r \\ \frac{3}{2}-\frac{3}{2}r & 1-r \end{bmatrix}.$$

Therefore, $x_{11} = (r, 2-r)$, $x_{12}(r) = (1+r, 4-2r)$, $x_{21} = (2r, 3-r)$ *and* $x_{22}(r) = (r, 2-r)$. *According to this fact that* $\underline{x}_i \leq \overline{x}_i$, $i = 1, 1$, *are monotonic decreasing functions then the solution* $x_{ij}(r) = (\underline{x}_{ij}(r), \overline{x}_{ij}(r))$ *for* $i, j = 1, 2$ *is a fuzzy solution.*

5 Conclusions

In this paper, we proposed a general model for solving a FLMEs. The original system (3.1) is replaced by two $n \times n$ crisp linear systems (3.4).

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