Common fixed point theorems of contractive mappings sequence in partially ordered G-metric spaces

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Received 3 February 2013; accepted 17 October 2013

Abstract

We consider the concept of $\Omega$-distance on a complete partially ordered G-metric space and prove some common fixed point theorems.

\textit{Key words:} $\Omega$-distance, fixed point, G-metric space

2010 AMS Mathematics Subject Classification : 47H10.

1 Introduction

The Banach fixed point theorem for contraction mapping has been generalized and extended in many direction [1-15]. Nieto and Rodriguez-

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Lopez [16], Ran and Reurings [17] and Petrusel and Rus [18] presented some new results for contractions in partially ordered metric spaces. The main idea in [12,16,17] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. Also, Mustafa and Sims [19] introduced the concept of G-metric. Some authors [20-24] have proved some fixed point theorems in these spaces. In [25] Gajić proved a common fixed point theorem for a sequence of mappings on this space. Recently, Saadati et al. [26], using the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of ω-distance due to Kada et al. [27]. In [28,29] some fixed point theorems proved and generalized under this concept.

In this paper, we extend some fixed point theorems by using this concept in partially ordered G-metric spaces.

At first we recall some definitions and lemmas. For more information see [19,26].

**Definition 1.1 [19]** Let $X$ be a non-empty set. A function $G : X \times X \times X \rightarrow [0, \infty)$ is called a G-metric if the following conditions are satisfied:

1. (i) $G(x, y, z) = 0$ if $x = y = z$ (coincidence),
2. (ii) $G(x, x, y) > 0$ for all $x, y \in X$, where $x \neq y$,
3. (iii) $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
4. (iv) $G(x, y, z) = G(p\{x, y, z\})$, where $p$ is a permutation of $x, y, z$ (symmetry),
5. (v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

**Definition 1.2 [19]** Let $(X, G)$ be a G-metric space,

1. (1) a sequence $\{x_n\}$ in $X$ is said to be G-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that for all
\( m, n, l \geq n_0, G(x_n, x_m, x_l) < \varepsilon \).

(2) A sequence \( \{x_n\} \) in \( X \) is said to be \( G \)-convergent to a point \( x \in X \) if, for each \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that for all \( m, n \geq n_0 \), \( G(x_m, x_n, x) < \varepsilon \).

**Definition 1.3** [19] Let \((X, G)\) be a \( G \)-metric space. Then a function \( \Omega : X \times X \times X \rightarrow [0, \infty) \) is called an \( \Omega \)-distance on \( X \) if the following conditions are satisfied:

(a) \( \Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z) \) for all \( x, y, z, a \in X \),
(b) for any \( x, y \in X, \Omega(x, y, .), \Omega(x, ., y) : X \rightarrow [0, \infty) \) are lower semi-continuous,
(c) for each \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( \Omega(x, a, a) \leq \delta \) and \( \Omega(a, y, z) \leq \delta \) imply \( G(x, y, z) \leq \varepsilon \).

**Example 1.1** [26] Let \((X, d)\) be a metric space and \( G : X^3 \rightarrow [0, \infty) \) defined by

\[
G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\},
\]

for all \( x, y, z \in X \). Then \( \Omega = G \) is an \( \Omega \)-distance on \( X \).

**Example 1.2** [26] In \( X = \mathbb{R} \) we consider the \( G \)-metric \( G \) defined by

\[
G(x, y, z) = \frac{1}{3}(|x - y| + |y - z| + |x - z|),
\]

for all \( x, y, z \in \mathbb{R} \). Then \( \Omega : \mathbb{R}^3 \rightarrow [0, \infty) \) defined by

\[
\Omega(x, y, z) = \frac{1}{3}(|x - y|) + |x - z|,
\]

for all \( x, y, z \in \mathbb{R} \) is an \( \Omega \)-distance on \( \mathbb{R} \).

For more examples see [26].

**Lemma 1.1** [26] Let \( X \) be a metric space with metric \( G \) and \( \Omega \) be
an $\Omega$-distance on $X$. Let $\{x_n\}, \{y_n\}$ be sequences in $X$, $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following:

1. If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$.
2. If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$, then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$.
3. If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then $\{x_n\}$ is a $G$-Cauchy sequence.
4. If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a $G$-Cauchy sequence.

**Definition 1.4** [26] $G$-metric space $X$ is said to be $\Omega$-bounded if there is a constant $M > 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, z \in X$.

## 2 Conclusion

In this section, we obtain common fixed point theorems for sequence of mappings satisfying contractiv and expansive conditions on partially ordered complete $G$-metric spaces.

**Definition 2.1** Suppose $(X, \leq)$ is a partially ordered space and $T : X \to X$ is a mapping of $X$ into itself. We say that $T$ is non-decreasing if for $x, y \in X$,

$$x \leq y \implies T(x) \leq T(y).$$

**Theorem 2.1** Let $(X, \leq)$ and $(Y, \leq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ and $Y$ such that $(X, G)$ and $(Y, G)$ are complete $G$-metric space and $\Omega_1$ is an $\Omega$-distance on $X$, $\Omega_2$ is $\Omega$-distance on $Y$ such that $X$ be $\Omega_1$-bounded and $Y$ be $\Omega_2$-bounded. Let $T_n : X \to Y$, $n \in \mathbb{N}$ and $S_n : Y \to X$, $n \in \mathbb{N} \cup \{0\}$ be a non-decreasing and continuous sequence of mappings with following properties:
(a) for all \( x, y, z \in X, x', y', z' \in Y \) and \( i, j, k \in \mathbb{N} \) such that \( 0 \leq r < 1 \),

\[
\Omega_1(S_iT_ix, S_jT_jy, S_kT_kz) \leq r \max \{\Omega_1(y, S_jT_jy, S_kT_kz), \Omega_1(x, y, z), \\
\Omega_2(T_iT_jy, T_kz)\},
\]

\[
\Omega_2(T_iS_{i-1}x', T_jS_{j-1}y', T_kS_{k-1}z') \leq r \max \{\Omega_2(y', T_jS_{j-1}y', T_kS_{k-1}z'), \\
\Omega_2(x', y', z'), \Omega_1(S_{i-1}x', S_{j-1}y', S_{k-1}z')\};
\]

(b) for every \( x, y, z \in X \) with \( y \neq S_nT_ny, n \in \mathbb{N} \),

\[
\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0;
\]

(c) for every \( x', y', z' \in Y \) with \( y' \neq T_nS_{n-1}y', n \in \mathbb{N} \),

\[
\inf\{\Omega(x', y', x') + \Omega(x', y', z') + \Omega(x', z', y') : x' \leq z'\} > 0;
\]

(d) \( \Omega_2(T_ix, T_jy, T_kz) = 0 \) for each \( x, y, z \in X \) and \( \Omega_1(S_ix', S_jy', S_kz') = 0 \) for each \( x', y', z' \in Y \).

Then \( \{S_nT_n\} \) has a unique common fixed point \( u \) in \( X \) and \( \{T_nS_{n-1}\} \)
has a unique common fixed point \( w \) in \( Y \). Furthermore, \( \lim_{n \to \infty} T_nu = w \) and \( \lim_{n \to \infty} S_nw = u \).

**Proof:** Let \( x_0 \in X \) such that \( S_nT_n(x_{n-1}) = x_n \) and \( T_n(x_{n-1}) = y_n \)
and \( x_n \leq x_{n+1} \) for any \( n \in \mathbb{N} \). For all \( n \in \mathbb{N} \) and \( t \geq 0 \),

\[
\Omega_1(x_n, x_{n+1}, x_{n+t}) = \Omega_1(S_nT_n(x_{n-1}), S_{n+1}T_{n+1}(x_n), S_{n+t}T_{n+t}(x_{n+t-1})) \leq r \max \{\Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_1(x_n, x_{n+1}, x_{n+t}), \\
\Omega_2(T_n(x_{n-1}), T_{n+1}(x_n), T_{n+t}(x_{n+t-1}))\} = r \max \{\Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_1(x_n, x_{n+1}, x_{n+t}), \\
\Omega_2(y_n, y_{n+1}, y_{n+t})\}.
\]

Then,

\[
\Omega_1(x_n, x_{n+1}, x_{n+t}) \leq r \max \{\Omega_1(x_{n-1}, x_n, x_{n+t-1}), \Omega_2(y_n, y_{n+1}, y_{n+t})\}.
\]
Similarly,
\[ \Omega_2(y_n, y_{n+1}, y_{n+t}) \leq r \max \{ \Omega_2(y_{n-1}, y_n, y_{n+t-1}), \Omega_1(x_{n-1}, x_n, x_{n+t-1}) \} \]
So,
\[ \Omega_1(x_n, x_{n+1}, x_{n+t}) \leq r^n \max \{ \Omega_1(x_0, x_1, x_t), \Omega_2(y_1, y_2, y_{t+1}) \}, \]
and
\[ \Omega_2(y_n, y_{n+1}, y_{n+t}) \leq r^n \max \{ \Omega_1(x_0, x_1, x_t), \Omega_2(y_0, y_1, y_t) \}. \]
Now, for any \( l > m > n \) with \( m = n + k \) and \( l = m + t \) \((k, t \in \mathbb{N})\), we have
\[ \lim_{n,m,l \to \infty} \Omega_1(x_n, x_m, x_l) = 0. \]
Since \( X \) is \( \Omega_1 \)-bounded and,
\[
\begin{align*}
\Omega_1(x_n, x_m, x_l) &\leq \Omega_1(x_n, x_{n+1}, x_{n+1}) + \Omega_1(x_{n+1}, x_m, x_l) \\
&\leq \Omega_1(x_n, x_{n+1}, x_{n+1}) + \Omega_1(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + \cdots + \Omega_1(x_{m-1}, x_m, x_l) \\
&\leq r^n M + r^{n+1} M + \cdots + r^{m-1} M \\
&\leq \sum_{j=0}^{n-m+1} r^{n-j} M \\
&\leq \frac{r^n}{1 - r} M.
\end{align*}
\]
So, by \( 0 \leq r < 1 \) and Part (3) of Lemma (1.6), \( \{x_n\} \) is a G-Cauchy sequence. Since \( X \) is G-complete, \( \{x_n\} \) converges to a point \( u \in X \). Similarly, \( \{y_n\} \) is a G-Cauchy sequence such that has a limit \( w \in Y \). Fixed \( n \in \mathbb{N} \) and by the lower semi-continuity of \( \Omega \), we have
\[
\begin{align*}
\Omega_1(x_n, x_m, u) &\leq \liminf_{p \to \infty} \Omega_1(x_n, x_m, x_p) \leq \frac{r^n}{1 - r} M, \quad m \geq n \\
\Omega_1(x_n, u, x_l) &\leq \liminf_{p \to \infty} \Omega_1(x_n, x_p, x_l) \leq \frac{r^n}{1 - r} M, \quad l \geq n.
\end{align*}
\]
Assume that \( u \neq S_n T_n u \). Since \( x_n \leq x_{n+1} \), we have
\[ 0 < \inf \{ \Omega_1(x_n, u, x_n) + \Omega_1(x_n, u, x_{n+1}) + \Omega_1(x_n, x_{n+1}, u) \} \]
\[ \leq 3 \inf \{ \frac{r^n}{1 - r} : n \in \mathbb{N} \} = 0, \]

which is a contraction. Therefore, \( u = S_nT_nu \) and consequently \( u \) is a common fixed point \( \{ S_nT_n \} \). Similarly, \( w \) is a common fixed point \( \{ T_nS_{n-1} \} \).

To prove the uniqueness, suppose \( \{ S_nT_n \} \) has another fixed point \( u' \). Then,
\[ \Omega_1(u, u', u') = \Omega_1(S_nT_nu, S_nT_nu', S_nT_nu') \leq r \max \{ \Omega_1(u, u', u'), \Omega_1(u', S_nT_nu', S_nT_nu') \} = r \max \{ \Omega_1(u, u', u'), \Omega_1(u', u', u'), \Omega_2(T_nu, T_nu', T_nu') \}. \]

By (d) either \( \Omega_1(u, u', u') = 0 \) or \( \Omega_1(u, u', u') \leq r \Omega_1(u', u', u') \).

Since,
\[ \Omega_1(u', u', u') = \Omega_1(S_nT_nu', S_nT_nu', S_nT_nu') \leq r \max \{ \Omega_1(u', u', u'), \Omega_1(u', S_nT_nu', S_nT_nu') \} \]
then, \( \Omega_1(u', u', u') = 0 \) and consequently \( \Omega_1(u, u', u') = 0 \). By Part (c) of Definition (1.3) fixed point of \( \{ S_nT_n \} \) is unique. Similarly, \( w \) is a unique fixed point of \( \{ T_nS_{n-1} \} \). By continuity of \( \{ T_n \} \), we have
\[ \lim_{n \to \infty} T_nu = \lim_{n \to \infty} T_n(x_{n-1}) = \lim_{n \to \infty} y_n = w. \]

Similarly, \( \lim_{n \to \infty} S_nw = u. \)

**Corollary 2.1** Let \((X, \leq)\) be a partially ordered space. Suppose that there exists a G-metric on \( X \) such that \((X, G)\) is a complete G-metric space and \( \Omega \) is an \( \Omega \)-distance on \( X \) such that \( X \) is \( \Omega \)-
bounded. Let \( T_n : X \rightarrow X \), \( n \in \mathbb{N} \) be a non-decreasing sequence of mappings with property that for any \( i, j, k \in \mathbb{N} \), we have:

(a) for all \( x, y, z \in X \) and \( 0 \leq r < 1 \),
\[
\Omega(T_i x, T_j y, T_k z) \leq r \max\{\Omega(x, y, z), \Omega(y, T_j y, T_k z)\};
\]
(b) for every \( x, y, z \in X \) with \( y \neq T_n y, n \in \mathbb{N} \),
\[
\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.
\]

Then \( \{T_n\} \) has a unique common fixed point \( u \) in \( X \) and \( \Omega(u, u, u) = 0 \).

**Proof:** It is sufficient that put \( \Omega = \Omega_1 = \Omega_2 \), \( X = Y \) and \( S_n = I_n \) that \( I_n \) is identity mapping on \( X \) in Theorem (2.2). \( \square \)

**Theorem 2.2** Let \((X, \leq)\) be a partially ordered space. Suppose that there exists a \( G \)-metric on \( X \) such that \((X, G)\) is a complete \( G \)-metric space and \( \Omega \) is an \( \Omega \)-distance on \( X \) such that \( X \) is \( \Omega \)-bounded. Let \( T_n : X \rightarrow X \), \( n \in \mathbb{N} \) be a non-decreasing sequence of mappings with property that for any \( i, j, k \in \mathbb{N} \), we have:

(a) for all \( x, y, z \in X \) and \( 0 \leq r < 1 \), \( \Omega(T_i x, T_j y, T_k z) \leq r \Omega(x, y, z) \);

(b) for every \( x, y, z \in X \) with \( y \neq T_n y, n \in \mathbb{N} \),
\[
\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.
\]

Then \( \{T_n\} \) has a unique common fixed point \( u \) in \( X \) and \( \Omega(u, u, u) = 0 \).

**Proof:** Theorem is proved by similar proof of Theorem 2.1. \( \square \)

**Corollary 2.2** Let \((X, \leq)\) be a partially ordered space. Suppose that there exists a \( G \)-metric on \( X \) such that \((X, G)\) is a complete \( G \)-metric space and \( \Omega \) is an \( \Omega \)-distance on \( X \) such that \( X \) is \( \Omega \)-bounded.
bounded. Let \( T_n : X \rightarrow X \), \( n \in \mathbb{N} \) be a non-decreasing sequence of mappings with property that for some \( m \in \mathbb{N} \) and each \( i,j,k \in \mathbb{N} \), we have:

(a) for all \( x,y,z \in X \) and \( 0 \leq r < 1 \), \( \Omega(T_i^m x, T_j^m y, T_k^m z) \leq r \Omega(x,y,z) \);

(b) for every \( x,y,z \in X \) with \( y \neq T_n y, n \in \mathbb{N} \),

\[
\inf \{ \Omega(x,y,x) + \Omega(x,y,z) + \Omega(x,z,y) : x \leq z \} > 0.
\]

Then \( \{T_n\} \) has a unique common fixed point \( u \) in \( X \) and \( \Omega(u,u,u) = 0 \).

**Proof:** By Theorem 2.2, the sequence \( \{T_n^m\} \) has the unique common fixed point \( u \). But,

\[
T_n u = T_n(T_n^m u) = T_n^{m+1} u = T_n^m(T_n u).
\]

So, \( T_n u \) is the fixed point \( \{T_n^m\} \). Now, by uniqueness of the fixed point, \( T_n u = u \). \( \square \)

**Definition 2.2** Let \( (X, G) \) be a \( G \)-metric space, \( \Omega \) be an \( \Omega \)-distance on \( X \) and \( T \) be a selfmapping on \( X \). Then \( T \) is called expansive mapping with respect \( \Omega \) if there exists a constant \( a > 1 \) such that for all \( x,y,z \in X \), we have:

\[
\Omega(Tx,Ty,Tz) \geq a\Omega(x,y,z).
\]

**Theorem 2.3** Let \( (X, \leq) \) be a partially ordered space. Suppose that there exists a \( G \)-metric on \( X \) such that \( (X, G) \) is a complete \( G \)-metric space and \( \Omega \) is an \( \Omega \)-distance on \( X \) such that \( X \) is \( \Omega \)-bounded. Let \( T_n : X \rightarrow X \), \( n \in \mathbb{N} \) be a non-decreasing sequence of surjective mappings and \( S_n : X \rightarrow X \), \( n \in \mathbb{N} \) be a non-decreasing sequence of injective mappings with property that for any \( i,j,k \in \mathbb{N} \), we have:

(a) for all \( x,y,z \in X \) and \( a > 1 \), \( \Omega(T_ix,T_jy,T_kz) \geq a\Omega(S_ix,S_jy,S_kz) \);
(b) for all \( n \in \mathbb{N} \), \( T_n \) and \( S_n \) commute;

(c) for every \( x, y, z \in X \) with \( y \neq T_n y, n \in \mathbb{N} \),

\[
\inf \{ \Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z \} > 0
\]

Then \( \{ T_n \} \) and \( \{ S_n \} \) have a unique common fixed point \( u \) in \( X \) and \( \Omega(u, u, u) = 0 \).

**Proof:** If \( T_i x = T_j y \) for any \( i \in \mathbb{N} \) and \( x, y \in X \), then,

\[
\Omega(T_i x, T_j y, T_j y) \geq a\Omega(S_i x, S_j y, S_j y);
\]

\[
\Omega(T_j y, T_i x, T_i y) \geq a\Omega(S_j y, S_i x, S_i y);
\]

thus,

\[
\Omega(S_i x, S_j y, S_j y) \leq \frac{1}{a}\Omega(T_i x, T_j y, T_j y);
\]

\[
\Omega(S_j y, S_i x, S_i y) \leq \frac{1}{a}\Omega(T_j y, T_i x, T_i y).
\]

Now, since \( a > 1 \) and \( X \) is \( \Omega \)-bounded then, for any \( \varepsilon > 0 \), we choose \( \delta = \frac{1}{a} M \), which implies, \( \Omega(S_i x, S_j y, S_j y) \leq \delta \) and \( \Omega(S_j y, S_i x, S_i y) \leq \delta \). By Part (c) of Definition (1.3), \( G(S_i x, S_j y, S_j y) \leq \varepsilon \). Since \( \varepsilon \) is arbitrary, hence \( S_i x = S_j y \). Now, by injectivity \( S_i \) for every \( i \in \mathbb{N} \), we imply that \( x = y \). So, \( T_n \) is injective and consequently invertible.

Let \( H_n \) be the inverse mapping of \( T_n \) for any \( n \in \mathbb{N} \). Then,

\[
\Omega(x, y, z) = \Omega(T_i (H_i x), T_j (H_j y), T_k (H_k z))
\]

\[
\geq a\Omega(S_i (H_i x), S_j (H_j y), S_k (H_k z)).
\]

So, for each \( x, y, z \in X \) and any \( i, j, k \in \mathbb{N} \), we obtain

\[
\Omega(S_i o H_i x, S_j o H_j y, S_k o H_k z) \leq r\Omega(x, y, z),
\]

where \( r = \frac{1}{a} \). Then \( \Omega(G_i x, G_j y, G_k z) \leq r\Omega(x, y, z) \), where \( G_n = S_n o H_n \). By Theorem 2.1, \( G_n \) or \( S_n o H_n \) have a unique common fixed point \( u \) in \( X \), i.e. \( G_n u = u = S_n o H_n u \). It follows that \( T_n(S_n(H_n u) = \)
Since, \( T_n \) and \( S_n \) commute, we obtain
\[
S_n(T_n(H_n u)) = T_n u \implies S_n u = T_n u,
\]
for any \( n \in \mathbb{N} \). If we put \( x = u \), \( y = H_j u \) and \( z = H_k u \), we have
\[
\Omega(T_i u, T_j(H_j u), T_k(H_k u)) \geq a \Omega(S_i u, S_j(H_j u), S_k(H_k u)).
\]
So,
\[
\Omega(T_i u, u, u) \geq a \Omega(S_i u, u, u) = a \Omega(T_i u, u, u).
\]
Since \( a > 1 \), then \( \Omega(T_i u, u, u) = 0 \). By putting \( x = H_j u \), \( y = H_j u \), \( z = H_k u \) and similar proof \( \Omega(u, u, u) = 0 \). Now by Part (3) of Definition (1.3), \( T_1 u = u \). Hence \( T_n u = S_n u = u \) and \( u \) is a unique common fixed point of \( T_n \) and \( S_n \).

The following corollary is a generalization of [18, theorem 2.1].

**Corollary 2.3** Let \( (X, \leq) \) be a partially ordered space. Suppose that there exists a G-metric on \( X \) such that \( (X, G) \) is a complete G-metric space and \( \Omega \) is an \( \Omega \)-distance on \( X \) such that \( X \) is \( \Omega \)-bounded. Let \( T_n : X \rightarrow X, n \in \mathbb{N} \) be a non-decreasing sequence of surjective mappings with property that for any \( i, j, k \in \mathbb{N} \), we have:

(a) for all \( x, y, z \in X \) and \( a > 1 \), \( \Omega(T_i x, T_j y, T_k z) \geq a \Omega(x, y, z) \);

(b) for every \( x, y, z \in X \) with \( y \neq T_n y, n \in \mathbb{N} \),
\[
\inf \{ \Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z \} > 0.
\]

Then \( \{ T_n \} \) has a unique common fixed point \( u \) in \( X \) and \( \Omega(u, u, u) = 0 \).

**Proof:** Follows from Theorem 2.3, by taking \( S_n = I_n \) for any \( n \in \mathbb{N} \) such that \( I_n \) is identity mapping on \( X \). □

**Corollary 2.4** Let \( (X, \leq) \) be a partially ordered space. Suppose that there exists a G-metric on \( X \) such that \( (X, G) \) is a complete
G-metric space and $\Omega$ is an $\Omega$-distance on $X$ such that $X$ is $\Omega$-bounded. Let $T_n : X \rightarrow X$, $n \in \mathbb{N}$ be a non-decreasing sequence of surjective mappings with property that for each $i, j, k \in \mathbb{N}$, we have:

(a) for all $x, y, z \in X$ and $a > 1$,

$$\Omega(T_i x, T_j y, T_k z) \geq a \max\{\Omega(x, y, y) + \Omega(y, y, z), \Omega(x, z, z) + \Omega(z, y, z)\},$$

(b) for every $x, y, z \in X$ with $y \neq T_n y$, $n \in \mathbb{N}$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, z) + \Omega(x, z, y) : x \leq z\} > 0.$$

Then $\{T_n\}$ has a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$.

**Proof:** Since by Part (a) of Definition (1.3),

$$a \max\{\Omega(x, y, y) + \Omega(y, y, z), \Omega(x, z, z) + \Omega(z, y, z)\} \geq a \Omega(x, y, z).$$

So, Theorem 2.3 implies that $\{T_n\}$ has a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$. $\Box$

**Acknowledgements**

The author would like to thank the referees for giving valuable comments and suggestions for the improvement of this paper.

**References**


