



Application of variational iteration method for solving singular two point boundary value problems

Sh. Sadigh Behzadi ^{a,*}

^a*Department of Mathematics, Islamic Azad University, Qazvin Branch,
Qazvin, Iran.*

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Abstract

In this paper, He's highly prolific variational iteration method is applied effectively for showing the existence, uniqueness and solving a class of singular second order two point boundary value problems. The process of finding solution involves generation of a sequence of appropriate and approximate iterative solution function equally likely to converge to the exact solution of the given problem which being processed out and improvised on its own at every step recursively. Moreover, Illustrative examples available to the context in literature when treated with, by application of such proposed method fetch encouraging results so as to justify and reveal its efficiency and usefulness of the method.

Key words: Variational iteration method, Transformation, Singular boundary value problem, Lagrange multiplier, Smooth function.

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* Corresponding author's E-mail: shadan_behzadi@yahoo.com(Sh.S.Behzadi)

1 Introduction

Several physical phenomenon prevailing in vast fields of applied sciences, mathematical physics, engineering and technology, medical sciences and astrophysics covering stellar structure, isothermal gas spheres, oxygen uptake kinetics including vital energy flow in human head and tumour growth inside body of a living being [1,3,9,25,29,36] when modeled, give rise in general to second order singular boundary value problems of type

$$\begin{aligned}(x^\alpha y(x)')' &= f(x, y(x)), \quad 0 < x \leq 1, \\ y(0) &= A, \quad y(1) = B.\end{aligned}\tag{1.1}$$

A, B are constants and $\alpha \in IR$. Out of such class, consider and confine to, on a subclass $\alpha \in (0, 1)$. The function $f(x, y(x))$ is a well defined real valued continuous function of two variables such that $\frac{\partial f}{\partial y}$ exists, is nonnegative and continuous on a domain $R = \{(x, y) : (x, y) \in [0, 1] \times IR\}$. Solution to such problems can also be ascertained to exist [11,27,28,30]. The problem (1.1) belonging to a specific area of differential equation has been a matter of immense research and keen interest to authors in recent past [7, 10, 22, 23, 35] vindicating its importance and scope. Variational iteration method is a modified Lagrange method [20] originally proposed by He [15-19] is a promising methodology of research into the various disciplines of science as an alternative to the method of linearization, transformation and discretization that have been earlier on applied over to such kind of problems.

2 Description of VIM

Consider any general differential equation involving a differential operator D .

$$Dy(x) = g(x), \quad x \in I \subseteq IR,\tag{2.1}$$

$y(x)$ being a sufficient smooth function on some domain and $g(x)$ an inhomogeneous function. Decompose Eq.(2.1) as

$$Ly(x) + N(y(x)) = g(x), \quad x \in I. \quad (2.2)$$

Where L and N are linear and nonlinear differential operators respectively.

Now, VIM corresponding to Eq.(2.2) generates a sequence of correction functional as

$$y_{n+1}(x) = y_n(x) + \int_0^x \mu(s)(Ly_n(s) + N\tilde{y}_n(s) - g(s)) ds, \quad n \geq 0. \quad (2.3)$$

Where μ is Lagrange multiplier determined optimally satisfying all stationary conditions after the variational method is applied on to Eq.(2.3). The importance and the very utility of method is endowed with the choice of assumption of considering even highly nonlinear and complicated dependent variables as restricted variables thereby synchronizing the error occurring due to process of finding solution to problem (1.1) to its minimum magnitude. \tilde{y}_n is the restricted variation i.e. $\delta\tilde{y}_n = 0$. Eventually, after μ is determined as desired, a proper selective function, may it be a linear or otherwise with respect to Eq.(2.2) is assumed as an initial approximation for finding next successive iterative function by Eq.(2.3) recursively. Thereafter given boundary conditions are imposed on the final or preferably on limiting value of sequence of iterative functional defined by Eq.(2.3) to procure reasonably effective satisfying solution to the problem (1.1).

3 Identification of Lagrange Multiplier by variational Method

The correction functional to (1.1) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \mu(s)(s^\alpha y_n'(s))' - \tilde{f}(s, y_n(s)) ds, \quad n \geq 0. \quad (3.1)$$

Where optimal value of $\mu(s)$ is identified naturally if variation with respect to $y_n(x)$ and restricted variation $\delta \tilde{y}_n(x) = 0$ is considered. Cosequently, from Eq.(3.1) we have

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \mu(s)(s^\alpha y_n'(s))' - \tilde{f}(s, y_n(s)) ds, \quad n \geq 0. \quad (3.2)$$

This simplifies to give

$$\delta y_n(x) = (1 - \mu'(s)) - \delta y_n(x) + \delta(\mu(s)s^\alpha y_n'(s)) |_{s=x} + \int_0^x (\mu'(s)s^\alpha)' \delta y_n(s) ds, \quad n \geq 0$$

Interalia, the stationary conditions are $1 - \mu'(s)s^\alpha = 0$, $\mu(x) = 0$, $(\mu'(s)s^\alpha)' = 0$ and that imply together to give

$$\mu(s) = \frac{s^{1-\alpha} - x^{1-\alpha}}{1 - \alpha}. \quad (3.3)$$

Therefore, from Eq.(3.1) the sequence of correction functional is given by

$$y_{n+1}(x) = y_n(x) + \frac{1}{1 - \alpha} \int_0^x (s^\alpha - x^\alpha)((s^\alpha y_n(s))' - \tilde{f}(s, y_n(s))) ds, \quad n \geq 0. \quad (3.4)$$

We may deduce from Eq.(3.4) that the limit of the convergent sequence $\{y_n(x)\}_{n=1}^\infty$ satisfying given boundary conditions is the exact solution of (1.1).

4 Convergence Analysis

In order to carry out convergence analysis of VIM with respect to given class (1.1) we consider Eq.(3.1) and observe that

$$y_{n+1}(x) = y_n(x) + \sum_{k=0}^{n-1} (y_{k+1}(x) - y_k(x)) \quad (4.1)$$

And that convergence of series Eq.(4.1) necessarily implies the convergence of sequence $\{y_n(x)\}_{n=1}^{\infty}$ of partial sums of the series Eq.(4.1).

Suppose $y_0(x)$ is assumed initial selective function then first successive variational iterate is

$$y_1(x) = \int_0^x \mu(s) ((s^\alpha y_0'(s))' - f(s, y_0(s))) ds. \quad (4.2)$$

Integrating by parts and then applying the stationary conditions, we deduce that

$$|y_1(x) - y_0(x)| = \left| \int_0^x (y_0'(s) + \mu(s)f(s, y_0(s))) ds \right|. \quad (4.3)$$

Or,

$$|y_1(x) - y_0(x)| \leq \int_0^x (|y_0'(s)| + |\mu(s)| |f(s, y_0(s))|) ds. \quad (4.4)$$

Again adopting similar procedures as in Eq.(4.2) and using stationary conditions likewise, relation Eq.(3.4) imply that

$$|y_2(x) - y_1(x)| = \left| \int_0^x (\mu(s)f(s, y_1(s)) - f(s, y_0(s))) ds \right|. \quad (4.5)$$

Or,

$$|y_2(x) - y_1(x)| \leq \int_0^x (|\mu(s)| |f(s, y_1(s)) - f(s, y_0(s))|) ds. \quad (4.6)$$

And, above all

$$|y_{n+1}(x) - y_n(x)| = \left| \int_0^x (\mu(s)f(s, y_n(s)) - f(s, y_{n-1}(s))) ds \right|. \quad (4.7)$$

Or,

$$|y_{n+1}(x) - y_n(x)| \leq \int_0^x (|\mu(s)| |f(s, y_n(s)) - f(s, y_{n-1}(s))|) ds, \quad n \geq 2. \quad (4.8)$$

Since $f(x, y)$ and $\frac{\partial f(x, y)}{\partial y}$ are continuous on R , therefore for fix $s \in [0, 1]$ and by virtue of, Mean value theorem $\exists(s, \theta_n^0(s)) \in R$ satisfying (say, $y_{n-1}(s) < \theta_n^0(s) < y_n(s)$), $\forall n \in \mathbb{N}$, $s \leq x \leq 1$ such that

$$|f(s, y_n(s)) - f(s, y_{n-1}(s))| = \left| \frac{\partial f(s, \theta_{n+1}^0(s))}{\partial y} \right| |y_n(s) - y_{n-1}(s)|, \quad \forall n \geq 2. \quad (4.9)$$

Now, suppose

$$M_1 = \sup(|y_0'(s)| + |\mu(s)| |f(s, y_0(s))|), \quad (4.10)$$

for

$$s \leq x \leq 1,$$

and

$$M_2 = \sup(|\mu(s)| \left| \frac{\partial f(s, \theta_n^0(s))}{\partial y} \right|), \quad (4.11)$$

for

$$s \leq x \leq 1, \quad n \in \mathbb{N}.$$

Again, choose

$$M = \sup(M_1, M_2). \quad (4.12)$$

Thereby to establish the truthfulness of the inequality

$$|y_{n+1}(s) - y_n(s)| \leq \frac{M^{n+1}x^{n+1}}{(n+1)!}, \quad \forall n \in \mathbb{N}. \quad (4.13)$$

Now, Eq.(??), Eq.(4.9), Eq.(4.10) and Eq.(4.12) together imply that

$$|y_1(x) - y_0(x)| \leq \int_0^x M_1 ds \leq \int_0^x M ds = Mx. \quad (4.14)$$

As well as,

$$|y_2(x) - y_1(x)| \leq \sup |\mu(s)| \left| \frac{\partial f(s, \theta_1^0(s))}{\partial y} \right| \int_0^x |y_1(s) - y_0(s)| ds,$$

or,

$$\begin{aligned} |y_2(x) - y_1(x)| &\leq \sup |\mu(s)| \left| \frac{\partial f(s, \theta_1^0(s))}{\partial y} \right| \int_0^x |y_1(s) - y_0(s)| ds \\ &= M \int_0^x M s ds, \quad s \leq x \leq 1, \quad n \in \mathbb{N} = \frac{M^2 x^2}{2}. \end{aligned}$$

Thus, the statement Eq.(4.13) is true for natural number $n = 1$.

Now, suppose that $|y_n(s) - y_{n-1}(s)| \leq \frac{M^n x^n}{n!}$ holds for some, $n \in \mathbb{N}$.

Then, again Eq.(4.8), Eq.(4.9) and Eq.(4.12) imply that

$$\begin{aligned} &|y_{n+1}(x) - y_n(x)| \\ &\leq \int_0^x |\mu(s)| \left| \frac{\partial f(s, \theta_n^0(s))}{\partial y} \right| |y_n(s) - y_{n-1}(s)| ds \\ &\leq \int_0^x |\sup(\mu(s))| \left(\sup \left| \frac{\partial f(s, \theta_n^0(s))}{\partial y} \right| \right) |y_n(s) - y_{n-1}(s)| ds, \quad s \leq x \leq 1, \quad n \in \mathbb{N}, \end{aligned}$$

or,

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\leq \sup(|\mu(s)|) \left| \frac{\partial f(s, \theta_{n+1}^0(s))}{\partial y} \right| \int_0^x |y_n(s) - y_{n-1}(s)| ds \\ &\leq M \int_0^x \frac{M^n x^n}{n!} ds = \frac{M^{n+1} x^{n+1}}{(n+1)!}. \end{aligned}$$

Therefore, by Principle of Induction

$$|y_{n+1}(x) - y_n(x)| \leq \frac{M^{n+1} x^{n+1}}{(n+1)!} \text{ holds } \forall x \in [0, 1] \text{ and } \forall n \in \mathbb{N}.$$

So the series Eq.(4.1) converges both absolutely and uniformly for all $x \in [0, 1]$. Since,

$$|y_0(x)| + \sum_{n=0}^{\infty} |y_{n+1}(x) - y_n(x)| \leq |y_0(x)| + \sum_{n=0}^{\infty} \frac{M^{n+1} x^{n+1}}{(n+1)!} = |y_0(x)| + (e^{Mx} - 1), \quad \forall x \in [0, 1].$$

Asserting that the series $y_0(x) + \sum_{k=0}^{\infty} (y_{k+1}(x) - y_k(x))$ converges uniformly $\forall x \in [0, 1]$ and hence the sequence of its partial sums

$\{y_n(x)\}_{n=0}^{\infty}$ converges to a limit function as a solution to (1.1) satisfying the given boundary condition.

5 Application

To begin with implementation and analyze scope of VIM we consider and apply this very method to find the solution of linear and nonlinear problems often referred, discussed and solved by different methods in literature. Specifically to mention is the method to solve it numerically and via numerical finite difference technique of solution.

Example 5.1

$$(x^\alpha y'(x))' = \beta x^{\alpha+\beta-2}((\alpha+\beta-1)+\beta x^\beta)y(x), \quad y(0) = 1, \quad y(1) = \exp(1). \quad (5.1)$$

Solution

The correction functional is

$$y_{n+1}(x) = y_n(x) + \int_0^x \mu(s)((s^\alpha y_n')' - \beta(\alpha+\beta-s^{\alpha+\beta-2}-\beta^2 s^{\alpha+\beta-2})y_n(s)) ds. \quad (5.2)$$

Inserting, $y(0) = y_0(x) = 1$ to Eq.(5.2) as selective initial approximation function we process out following induced successive iterative approximate solutions

$$\begin{aligned} y_1(x) &= 1 + x^\beta + \beta \frac{x^{2\beta}}{2(\alpha+\beta-1)}, \\ y_2(x) &= 1 + x^\beta + \frac{x^{2\beta}}{2.1} + \beta \frac{x^{3\beta}}{3(\alpha+3\beta-1)}, \\ y_3(x) &= 1 + x^\beta + \frac{x^{2\beta}}{2.1} + \frac{x^{3\beta}}{3.2.1} + \beta \frac{x^{4\beta}}{4.2(\alpha+4\beta-1)}, \\ y_4(x) &= 1 + x^\beta + \frac{x^{2\beta}}{2.1} + \frac{x^{3\beta}}{3.2.1} + \frac{x^{4\beta}}{4.3.2.1} + \beta \frac{x^{5\beta}}{5.3.2(\alpha+5\beta-1)}, \\ y_5(x) &= 1 + x^\beta + \frac{x^{2\beta}}{2.1} + \frac{x^{3\beta}}{3.2.1} + \frac{x^{4\beta}}{4.3.2.1} + \frac{x^{5\beta}}{5.4.3.2.1} + \beta \frac{x^{6\beta}}{6.4.3.2(\alpha+6\beta-1)}. \end{aligned}$$

Similarly, continuing in like manner inductively we find the general

term of the sequence

$$\begin{aligned} y_n() &= 1 + x^\beta + \frac{x^{2\beta}}{2.1} + \frac{x^{3\beta}}{3.2.1} + \frac{x^{4\beta}}{4.3.2.1} + \dots + \frac{x^{n\beta}}{n!}, \\ y_n(x) &= \sum_{k=0}^n \frac{x^{k\beta}}{k!} + \frac{n\beta x^{(n+1)\beta}}{(n+1)!(\alpha+(n+1)\beta-1)}. \end{aligned} \quad (5.3)$$

Hitherto, we observe that $T_n = \frac{n\beta x^{(n+1)\beta}}{(n+1)!(\alpha+(n+1)\beta-1)}$ (say), is the general term of a convergent Series $\sum_{n=0}^{\infty} \frac{n\beta x^{(n+1)\beta}}{(n+1)!(\alpha+(n+1)\beta-1)}$.

Therefore, $\lim(n \rightarrow \infty) \frac{n\beta x^{(n+1)\beta}}{(n+1)!(\alpha+(n+1)\beta-1)} = 0$.

Now the relation Eq.(5.3) facilitates $y(x) = \lim(n \rightarrow \infty) (\sum_{k=0}^n \frac{x^{k\beta}}{k!}) = \exp(x^\beta)$.

That produces exact solution to problem Eq.(5.1)

Example 5.2

$$\begin{aligned} (x^\alpha y'(x))' &= \frac{\beta x^\alpha}{4+x^\beta} (\beta x^\beta e^y - (\alpha + \beta - 1)), \\ y(0) &= \ln \frac{1}{4}, \quad y(1) = \ln \frac{1}{5}. \end{aligned} \quad (5.4)$$

Solution

Let, $y_0 = y(0) = \ln \frac{1}{4}$, be the selective initial approximation function . Then by VIM,

First iterative approximate solution to Eq.(5.4) simplifies to

$$y_1(x) = \ln \frac{1}{4} + \int_0^x \frac{\mu(s)}{4+x^\beta} \left(\frac{\beta^2 s^{\alpha+2\beta-2}}{4} - (\alpha + \beta - \beta^\alpha s^{\alpha+\beta-2}) \right) ds. \quad (5.5)$$

Whereas $\mu(s)$ is optimally identified Lagrange multiplier as existing in Eq.(3.3) and after simplifying Eq.(5.4) the required first approximate solution to Eq.(5.3) satisfying the given boundary condition

$y(0) = \ln \frac{1}{4}$ is as following,

$$y_1(x) = \ln \frac{1}{4} - \frac{x^\beta}{4} + \frac{1}{2} \left(\frac{x^\beta}{4}\right)^2 + \sum_{n=3}^{\infty} \left(\frac{\alpha + 2\beta - 1}{\alpha + n\beta - 1}\right) \left(\frac{-1}{n}\right) \left(\frac{x^\beta}{4}\right)^n. \quad (5.6)$$

Now, if the boundary condition $y(1) = \ln \frac{1}{5}$ expressed in expanded real series form and imposed on $y_1(x)$ is matched. Then $y_1(x)$ satisfies exactly the problem equation if the terms $\left(\frac{\alpha+2\beta-1}{\alpha+n\beta-1}\right)$ and $\left(\frac{-1}{n}\right) \left(\frac{x^\beta}{4}\right)^n$ are treated independent to each other and arbitrary parameter β is allowed to approach zero in $\left(\frac{\alpha+2\beta-1}{\alpha+n\beta-1}\right)$ independently. With this, even the first iterate may produce exact solution $y(x)$ to Eq.(5.3).

$$y(x) = y_1(x) = \ln \frac{1}{4} - \frac{x^\beta}{4} + \frac{1}{2} \left(\frac{x^\beta}{4}\right)^2 + \sum_{n=3}^{\infty} \left(\frac{-1}{n}\right) \left(\frac{x^\beta}{4}\right)^n = \ln \frac{1}{4 + x^\beta}.$$

6 Conclusion

This is to mention that variation iteration method successfully applies when applied to a linear as well as a nonlinear boundary value problem of the considered class. Side by side a separate convergence analysis to the proposed method is also been presented and discussed in a lucid and exhaustive manner. Frontier examples are being solved that focus and assert on proper selection of selective function which on a careful imposition of boundary condition could lead to an exact solution or any other solution of high accuracy even to a non-linear problem just by performing only some simplifications.

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